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MATH-S-400
MATHEMATICS AND ECONOMIC MODELLING

Contents

Logic and proofs

A teacher announces in class that an examination will be held on some day during the following week, and moreover that the examination will be a surprise. The students, having followed a course in mathematical logic argue that a surprise exam cannot occur. For suppose the exam were on the last day of the week. Then on the previous night, the students would be able to predict that the exam would occur on the following day, and the exam would not be a surprise. So it is impossible for a surprise exam to occur on the last day. But then a surprise exam cannot occur on the penultimate day, either, for in that case the students, knowing that the last day is an impossible day for a surprise exam, would be able to predict on the night before the exam that the exam would occur on the following day. Similarly, the students argue that a surprise exam cannot occur on any other day of the week either. Confident in this conclusion, they decide not to study for the exam. The next week, the teacher gives an exam on Wednesday, which, despite all the above, was an utter surprise to all students.

Timothy Y. Chow, The American Mathematical Monthly, Vol. 105, No. 1 (Jan., 1998)

Logic is the language of reasoning. It deals with the evaluation of arguments. Its main aim is to develop a system of methods and principles that can be used as a criteria for evaluating the validity of certain chains of arguments. Propositional logic, which is a subfield of logic, is a logical system built around two values *True* and *False*, called the truth values. For simplicity we can assign the value 1 to something that is *True* and the value 0 to something that is *False*.[1] The aim of propositional logic is to find out the truth value of formula's.

[1] We don't allow for statements that can be both true and false or for statements that are neither true nor false.

If p is a formula, and this formula is true, we write

$$T(p) = 1.$$

If the formula is false, we write

$$T(p) = 0.$$

The main objective of mathematics is to find (non trivial) formula's that are true. Many formulas are obtained by putting together or transforming other formulas. In order to do this, however, we need to give the rules that determine how various formulas can be combined into other ones and how their truth value is determined. Also, as it is impossible to get something from nothing, we need some basic building blocks. These building blocks are what we call **atomic statements**.

AN ATOMIC STATEMENT, which we represent by the letters $p, q, r, s, \ldots$, are the basic building blocks of a formula. Atomic statements are either *True* or *False*. Every atomic statement is also a formula.

The following gives some examples of atomic statements:

- $p =$ "it is snowing"
- $q =$ "all consumers are rational"
- $r =$ "tax cuts increase welfare"
- $s =$ "mathematics is boring"

NEW FORMULAS ARE build up from other formula's. A first way to transform a given formula into another formula is by **negation**. If p is a formula then, $\neg p$ is also a formula. Intuitively, $\neg p$ means

*"It is **not** the case that p".*

Of course p is true if and only if $\neg p$ is false, as such, we obtain the rule

$$T(\neg p) = 1 - T(p).$$

It is possible to represent this in a so called truth table. The first column of the truth table enumerates all possible values of the formula p. The second column enumerates the corresponding values of $\neg p$.

Examples of negation are,

- $\neg p =$ "it is **not** snowing."
- $\neg q =$ "**not** all consumers are rational."
- $\neg r =$ "a tax cut does **not** increases welfare."
- $\neg s =$ "mathematics is **not boring."**

p	$\neg p$	$\neg(\neg p)$
1	0	1
0	1	0

Truth tables are useful because they allow us to derive equivalence between two expressions. The equivalence between the first and third column for example shows that $p \equiv \neg(\neg p)$. For every possible truth value of p, p has the same truth value as $\neg(\neg(p)))$.[2]

[2] In contrast to the English language, mathematics has not problem with double negations.

NEGATION TRANSFORMS ONE formula into another one. On the other hand, it is also possible to combine two existing formulas into a new one. The first rule that does this is **conjunction**. If p and q are two formulas then $p \wedge q$ is also a formula and it means

*"it is the case that p **and** q".*

For example,

- $p \wedge q$ = "it is snowing **and** all consumers are rational."

So the formula $p \wedge q$ is assigned the truth value 1 if and only if both the formula p is true and the formula q is true. In other words, $p \wedge q$ is false if at least one of them is false. The mathematical rule is,[3]

$$T(p \wedge q) = 1 \text{ if and only if } T(p) = 1 \text{ and } T(q) = 1.$$

[3] Or equivalently,

$$T(p \wedge q) = \min\{T(p), T(q)\}.$$

The following gives the truth table for $p \wedge q$. Here the first two columns give every possible combination of truth values for p and q. The third column gives the value of $p \wedge q$.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

THE SECOND RULE to combine two formulas into a new one is **disjunction**. If p and q are formula then $p \vee q$ is also a formula and it means that

*"it is the case that p **or** q"*

Here the "or" is inclusive, so either p is true, q is true or they are both true. The mathematical rule is that,[4]

$$T(p \vee q) = 1 \text{ if and only if } T(p) = 1 \text{ or } T(q) = 1.$$

For example,

- $p \vee q$ = "it is snowing **or** all consumers are rational."

[4] Or equivalently,

$$T(p \vee q) = \max\{T(p), T(q)\}.$$

The following truth table demonstrates that the $\vee$ rule can also be defined by combining $\wedge$ and $\neg$:

$$p \vee q \equiv \neg(\neg p \wedge \neg q).$$

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
1	1	1	0	0	0	1
1	0	1	0	1	0	1
0	1	1	1	0	0	1
0	0	0	1	1	1	0

Observe that for any possible value combination of p and q, the formula $p \vee q$ has the same truth value as $\neg(\neg p \wedge \neg q)$. Equivalently,[5]

$$\neg(p \vee q) \equiv (\neg p \wedge \neg q).$$

[5] Notice the similarity with De Morgan's law:

$$(A \cup B)^c = A^c \cap B^c.$$

A THIRD VERY important rule to combine two formula's into a new one is by **implication**. If p and q are formula then $p \to q$ is also a formula and it means,

*"**If** p, **then** q".*

Here $p \to q$ is true if q is true whenever p is true. The mathematical rule is,[6]

$$T(p \to q) = 1 \text{ if and only if } T(p) = 1 \text{ implies } T(q) = 1.$$

For example,

- $p \to q$ = "**If** it is snowing **then** all consumers are rational."

[6] Or equivalently,

$$T(p \to q) = \max\{1 - T(p), \min\{T(p), T(q)\}\}.$$

So for $p \to q$ to hold either p is *False*, in which case q can either be *True* or *False* or p is *True* but in that case, q must also be *True*.

p	q	$p \to q$	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
1	1	1	0	0	1
1	0	0	1	1	0
0	1	1	0	0	1
0	0	1	1	0	1

Above truth table shows that[7]

$$p \to q \equiv \neg(p \wedge \neg q).$$

This is the basic justification for a proof by contradiction. In order to show that p implies q you start by assuming that both p holds and q does not hold. Next you use this to derive a contradiction, which shows that $\neg(p \wedge \neg q)$ is *True*.

A FINAL RULE to combine two formula's into a third one is by **equivalence**. If p and q are formula then $p \leftrightarrow q$ is also a formula and it means,

$$\text{“}p \textbf{ if and only if } q\text{”}.$$

The mathematical rule is,[8]

$$T(p \leftrightarrow q) \text{ if and only if } \begin{pmatrix} (T(p) = 1 \text{ and } T(q) = 1), \\ \text{or ,} \\ (T(p) = 0 \text{ and } T(q) = 0) \end{pmatrix}.$$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

From this, we see that $p \leftrightarrow q$ has the same truth value as $(p \to q) \wedge (q \to p)$. As such, in order to show the equivalence between two statements you can show that the first implies the second and the second implies the first.

[7] Notice the equivalence between

$$A \subseteq B,$$

and

$$(A \cap B^c) = \emptyset.$$

Exercise: show that for any formula $p \vee \neg p$ is always *True*. This is called the law of the excluded middle. Some mathematicians actually contest this rule. If you don't agree with the law of the excluded middle, you can't use a proof by contradiction.

For example,

- $p \leftrightarrow q$ = "it is snowing **if and only if** all consumers are rational."

[8] Or equivalently,

$$T(p \leftrightarrow q) = \max \begin{Bmatrix} \min\{T(p), T(q)\}, \\ \min\{1 - T(p), 1 - T(q)\} \end{Bmatrix}.$$

Predicate logic

UNTIL NOW WE have studied what is called propositional logic. Predicate logic builds upon propositional logic by introducing two additional symbols, called **quantifiers**, namely the **existential** ($\exists$) and **universal** ($\forall$) quantifiers. These quantifiers must be combined with variables belonging to some particular set. Recall that,

- $\forall a \in A : P(a)$ means "for all a in A such that $P(a) \ldots$"
- $\exists a \in A : P(a)$ means "there exists an a in A such that $P(a) \ldots$"
- $\exists! a \in A : P(a)$ means " there exists a unique a in A such that $P(a) \ldots$"

Observe that the $\forall$ quantifier works as a long chain of $\wedge$ connectives over all elements in some set A. If $A = \{a_1, a_2, \ldots, a_n\}$ then ,

$$\forall a \in A : P(a)$$

is the same as,

$$(P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_n)) .$$

On the other hand, the $\exists$ operator works as a long chain of $\vee$ connectives.

$$\exists a \in A : P(a),$$

Is the same as,

$$(P(a_1) \vee P(a_2) \vee \ldots \vee P(a_n)) .$$

In addition, the quantifiers also work on sets that are of infinite size, which is the main reason for using them.

In order to prove statements containing quantifiers, it can be useful to know some propositions that are logical true (without going too formal). The following two rules shows you how to negate a formula containing quantifiers.[9]

- $\neg(\forall a \in A : P(a))$, is equivalent to $\exists a \in A : \neg P(a)$.
- $\neg(\exists a \in A : P(a))$, is equivalent to $\forall a \in A : \neg P(a)$.

Independent of what formula P is, these statements are always true. "Not (for all a, $P(a)$)" is equivalent as the statement "there exists an $a \in A$ such that not $P(a)$".

For example, let us negate

$$\forall x \forall y : ((x > 0) \wedge (y < 0)) \rightarrow (xy < 0)$$

First we change the $\forall$ quantifiers into an $\exists$ quantifier. Next, we negate the premises. This gives,

$$\exists x \exists y : ((x > 0) \wedge (y < 0) \wedge (xy \geq 0).$$

Next, we can also combine $\forall$ and $\exists$ quantifiers if we have propositions that use multiple variables

$$\forall a \in A, \exists b \in B : P(a, b).$$

There is no problem in exchanging te two quantifiers if they are of the same type,

$$\forall a, \forall b : P(a, b) \leftrightarrow \forall b, \forall a : P(a, b).$$

[9] Notice the similarity with the equivalence between

$$\neg(p \wedge q \wedge r),$$

and

$$(\neg p \vee \neg q \vee \neg r)$$

and the equivalence between

$$\neg(p \vee q \vee r),$$

and

$$(\neg p \wedge \neg q \wedge \neg r).$$

However, one should be very careful when the quantifiers are of different types. It is true that

$$\exists a \in A, \forall b \in B : P(a,b) \rightarrow \forall b \in B, \exists a \in A : P(a,b).$$

The opposite, however does not need to hold. The left hand side means that you have to select one a and combine it with any b to evaluate $P(a,b)$. As such, a should be independent of b. The right hand side means that for any b, you should find an a to evaluate $P(a,b)$. As such, the choice of a may depend on the choice of b.[10]

[10] The following example illustrates.

- $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z} : a \geq b$ is true but
- $\exists b \in \mathbb{Z}, \forall a \in \mathbb{Z} : a \geq b$ is false.

LET US NOW have a look at how the quantifiers interact with the logical connectives. Let P and Q be two propositions, then

$$\exists a \in A : (P(a) \vee Q(a)) \leftrightarrow [(\exists a \in A : P(a)) \vee (\exists a \in A : Q(a))].$$

If the left hand side is true, then there is an a such that $P(a)$ or $Q(a)$ is true. Of course the same a can be used on the right hand side to show $\rightarrow$. If the right hand side is true, then we know that either there is an a such that $P(a)$ holds or there is an a' such that $Q(a')$ holds. Use a or a' to make the left hand side valid.

The same equivalence result does not hold for the existential qualifier and the connective $\wedge$.

$$\exists a \in A : (P(a) \wedge Q(a)) \rightarrow [(\exists a \in A : P(a)) \wedge (\exists a \in A : Q(a))].$$

To go from left to right, we do the same as before, i.e. we use the a that makes the left hand side true. However, this does not work in the other way, since the a and a' that make the right hand side true may be different.[11]

[11] See also Exercise **??** for a similar results for the universal quantifier.

Proofs

ALL RESULTS IN mathematics are obtained via proofs. There are different kind of proofs according to the logical format that are used to construct the proof.

THE DIRECT PROOF of showing $p \rightarrow q$ is to find a set of 'intermediate' formula's $p_0, p_1, \ldots, p_n$ such that $p_i \rightarrow p_{i+1}$ for all $i = 0, \ldots, n-1$ and where $p_0 = p$, $p_n = q$. This gives a chain,

$$p \rightarrow p_1 \rightarrow p_2 \rightarrow \ldots \rightarrow q.$$

As such, if p is *True*, it then follows that q must also be *True*. Consider the following assertion, that we want to prove.

"For all integers x, if x is odd, then x^2 is also odd."

The direct proof starts with the assumption, p_0 = "x is odd" and creates a chain of intermediate 'truths' which end by sentence q = "x^2 is odd", we can use the following chain of reasoning,

p_0: x is odd

p_1: then, there is an integer n such that $x = 2n + 1$.[12]

p_2: then, $x^2 = (2n+1)^2$,

p_3: then, $x^2 = 4n^2 + 4n + 1$,[13]

p_4: then, $x^2 = 4(n^2 + n) + 1$,

p_5: then, $x^2 = 2z + 1$ where $z = 2(n^2 + n)$ is an integer,

q: then, x^2 is odd.[14]

[12] Every odd number can be written as two times a number plus one.

[13] Remember: $(a+b)^2 = a^2 + 2ab + b^2$.

[14] As x^2 is two times a number plus one.

A proof by contrapositive relies on the equivalence between $p \to q$ and $\neg q \to \neg p$, as illustrated in the following truth table.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
1	1	1	0	0	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

So, in order to show that $p \to q$ we can start with $\neg q$ and use a direct proof to show that $\neg p$. Let us demonstrate this by proving the following,

"If the sum of two integers $x + y$ is even then either both integers x and y are even or both integers x and y are odd."

A direct proof of this would be quite involved.[15] However, we can easily proof the assertion by starting from the assumption that x and y are not of the same parity.

[15] Try it.

$\neg q$: assume x and y don't have the same parity,

q_1: then, wlog assume x is odd and y is even,[16]

q_2: then, there are integers z and w such that $x = 2z + 1$ and $y = 2w$,

q_3: then $x + y = 2z + 2w + 1 = 2(z + w) + 1$,

$\neg p$: then, $x + y$ is odd.

[16] wlog is an abbreviation for 'without loss of generality'.

A PROOF BY CONTRADICTION relies on the equivalence between p and $\neg(\neg p)$. In order to show that p is true, one starts from the assumption that $\neg p$ is true. Then, you show that this leads to a contradiction,[17] showing indeed that $\neg(\neg p) \equiv p$ is true.

For an example, let's demonstrate that,

$$\text{“}\sqrt{2} \text{ is irrational.”}$$

A direct proof of this assertion would be very difficult. So let us proceed by contradiction.

$\neg p$: $\sqrt{2}$ is rational,

p_1: then, $\sqrt{2} = a/b$ where a and b are integers and not both even.[18]

p_2: then, $b\sqrt{2} = a$,

p_3: then, $b^2 2 = a^2$,

p_4: then, a^2 is even,

p_5: then, a is even,[19]

p_6: then, there is a number z such that $a = 2z$,

p_7: then, $a^2 = 4z^2$,

p_8: then, $b^2 = 2z^2$,

p_9: then, b^2 is even,

p_{10}: then, b is even.

From this we see that p_{10} and p_5 contradict p_1, So $\sqrt{2}$ is not rational which shows that it is irrational.

A proof by contradiction can also be used to demonstrate an implication $p \rightarrow q$. In this case, one starts from the assumption $\neg(p \rightarrow q)$ and show that this leads to a contradiction. In addition, $\neg(p \rightarrow q) \equiv (\neg q \wedge p)$ so one can start from the assumption that p and $\neg q$ are both true and use this to derive a contradiction.

[17] In other words, you show that $\neg p$ is false.

[18] Every rational number can be written as the ratio of two numbers that have no common divisors (except 1).

[19] Follows from the previous proof.

PROOF BY INDUCTION is somewhat different form the proofs presented above. It can be used if the proposition to be proven must hold for each value of a parameter that takes a value in the set of natural numbers, $\mathbb{N}_0$.

Assume that we want to proof that a formula $p(n)$ is true for all natural numbers n.[20] The proof by induction starts by proving that $p(1)$ holds. Then it goes on to show that $p(k) \rightarrow p(k+1)$ for all natural numbers k. From this, it can be concluded that $p(n)$ holds for all natural numbers n.

$$p(1) \rightarrow p(2) \rightarrow \ldots \rightarrow p(k) \rightarrow p(k+1) \rightarrow \ldots.$$

[20] There also exists a proof technique called transfinite induction. Here the induction of over all ordinals. These ordinals may contain numbers that are larger than infinite...

As an illustration, let us prove,

"For all natural numbers n: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."

First $p(1)$ requires us to demonstrate that $\sum_{i=1}^{1} i = 1 = 1(1+1)/2 = 1$, so this is true. For the induction step let us assume that $p(k)$ holds, i.e. $\sum_{i=1}^{k} i = k(k+1)/2$. We need to show that this implies

$$p(k+1) \equiv \sum_{i=1}^{k+1} i = (k+1)(k+2)/2.$$

$p(k)$: $\sum_{i=1}^{k} i = k(k+1)/2$,

$p_1(k)$: then, $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$,

$p_2(k)$: then, $\sum_{i=1}^{k+1} i = k(k+1)/2 + (k+1)$,[21]

$p_3(k)$: then, $\sum_{i=1}^{k+1} i = \frac{k(k+1)+2(k+1)}{2}$,

$p(k+1)$: then, $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

[21] By the induction hypothesis.

Be very careful with proving statements using quantifiers $\forall$ and $\exists$. For the universal quantifier you should check the proposition for any element of the set. So you will work with variables. For the existential quantifier, it could suffice to only give one example.

For example, in order to prove,

$$\forall a \in \mathbb{Z} : a^2 \geq 0,$$

we need to show that $a^2 \geq 0$ for all integers. One way to do this is to prove the statement for all non-negative numbers and prove the statement for all negative numbers. In order to prove,

$$\exists a \in \mathbb{Z} : a^2 = 9,$$

we only need to find one number for which the statement holds.[22]

[22] In this case we can pick either $a = 3$ or $a = -3$. However, we only need to find one of the two cases in order to establish the proof.

Reference

- Appendix A1.3 of Simon and Blume, (1994), Mathematics for Economists, W. W. Northon and Company, New York, London.

Exercises

1. Let p, q, and r be the following statements:
   - p: Traveling to Mars is expensive.
   - q: I will travel to Mars.

- r: I have money.

Express the following English sentences as symbolic expressions:

- I have no money and I will not travel to Mars.
- I have no money and travelling to Mars is expensive, or I will travel to Mars.
- It is not true that, I have money and will travel to Mars.
- Travelling to Mars is not expensive and I will go to Mars, or travelling to Mars is expensive and I will not go to Mars.

2. Construct truth tables for each of the following formulas.

- $P \wedge (Q \vee \neg P)$
- $\neg P \to Q$
- $Q \vee \neg(P \wedge Q)$
- $P \to \neg(P \wedge Q)$
- $P \to (P \wedge Q)$
- $Q \to (P \to Q)$
- $P \wedge \neg P$
- $(Q \vee P) \wedge \neg P$

3. There is a party tonight.

- John comes to the party if Mary or Ann comes.
- Ann comes to the party if Mary does not come.
- If Ann comes to the party, John does not.

Try to figure out who will come to the party.

4. For each of the following propositions, state the negation:

- $x > 0$ and $y > 0$
- All x satisfy $x \geq a$
- Neither x nor y is less than 5.
- For each $\varepsilon > 0$ there exists a $\delta > 0$ such that B is satisfied.
- $\forall \varepsilon > 0, \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, (m \geq n) \to (|x_m - a| < \varepsilon)$.

5. Prove the statement. If 6 is a prime number, then $6^2 = 30$.

6. Prove the statement: For all integers m and n, if m and n are odd integers, then $m + n$ is an even integer.

7. Prove the statement: For all integers m and n, if the product of m and n is even, then m is even or n is even.

8. Prove the statement: For all nonnegative real numbers a, b and c, if $a^2 + b^2 = c^2$, then $a + b \geq c$.

9. Prove the pigeonhole principle: If if k, n are integers and $kn + 1$ objects (pigeons) are distributed into n boxes (pigeon holes), then some box must contain at least $k + 1$ of the objects.

10. Use the pigeonhole principle above to show that if there are n people who can shake hands with one another (where $n > 1$), then there is always a pair of people who will shake hands with the same number of people.

11. Proof that there are infinitely many primes.

12. Show that for all $n \in \mathbb{N}$, $n < 2^n$.

13. Show that $\sum_{t=1}^{n} 2^t = 2^{n+1} - 2$.

14. The price of a stock is defined as the sum of the actualized values of the dividends. Suppose that the yearly dividend is K and the yearly interest rate is r. Prove that if we take n years into account, that the price of the stock is given by,

$$p = \frac{K}{r}\left(1 - \left(\frac{1}{1+r}\right)^n\right).$$

15. Show that there are two irrational numbers a and b such that a^b is rational.

Sequences and limits

We use standard notation as much as possible. The set $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ is the infinite set of strict positive integers $\{1, \ldots\}$. Elements of these sets are sometimes called scalars and they are denoted by small letters $x, y, z, \ldots$.

The set $\mathbb{R}^k$ is the k-fold Cartesian product of $\mathbb{R}$, elements of $\mathbb{R}^k$ are k-dimensional column vectors and they are denoted in bold, $\mathbf{x}, \mathbf{y}, \mathbf{z} \ldots$, e.g.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

We use subindices $+$ and $++$ to denote non-negative and strict positive parts of these subsets.[23] Corresponding row vectors are denoted by $\mathbf{x}'$,[24]

$$\mathbf{x}' = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

The dot product of a row and column vector is denoted by $\mathbf{x} \cdot \mathbf{y}$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{n=1}^{k} x_n y_n.$$

[23] For example $\mathbb{R}^k_{++}$ is the set of k-dimensional vectors whose components are all strictly positive while $\mathbb{R}_+$ is the set of vectors whose components are non-negative.

[24] A prime after a vector or matrix is used to denote the transpose.

Euclidean distance

Much of real analysis deals with limits of sequences and vectors. In order to discuss these notions, we need to have some way to measure the distance between two numbers and, more importantly, the distance between two points or vectors in k-dimensional Euclidean space. For scalars we use the absolute value of the difference between the numbers,

$$|x - y| = \begin{cases} x - y \text{ if } x \geq y, \\ y - x \text{ if } x < y \end{cases}$$

For vectors, we use the notion of an Euclidean distance. We use the notation $\|\mathbf{x}\|$ for the Euclidean norm. It is given by the following formula.

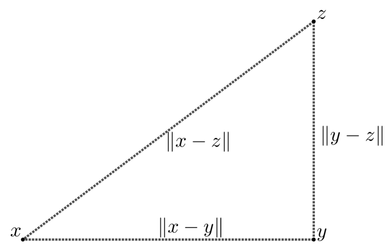
$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{k} x_i^2}$$

Notice that although $\mathbf{x}$ is a vector, its norm $\|\mathbf{x}\|$ is a number. The value of $\|\mathbf{x}\|$ is equal to the length of the line segment from the origin $\mathbf{0}$ to the point $\mathbf{x}$.[25] The (Euclidean) distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by $\|\mathbf{x} - \mathbf{y}\|$ and is defined as,

[25] The point $\mathbf{x}$ in $\mathbb{R}^k$ is given by the value of its coordinates.

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}.$$

Figure 1: Euclidean distance $\|\mathbf{x} - \mathbf{y}\|$.



The distance between the points (x_1, x_2) and (y_1, y_2) is given by the square root of the sum of the squares of the lengths of the sides of the right angled triangle.

In two dimensions, this gives

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

which can also be found by applying Pythagoras' rule to compute length of the line segment between the points (x_1, x_2) and (y_1, y_2).

The norm $\|.\|$ can be seen as the natural extension of Pathagoras' to settings with more than two dimensions.

The norm function $\|.\|$ satisfies the following desirable properties.[26]

1. $\|\mathbf{x} - \mathbf{y}\| \geq 0$ and $\|\mathbf{x} - \mathbf{y}\| = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
2. $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.
3. $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|$.

[26] A function ρ that satisfies the following conditions

- $\rho(\mathbf{x}, \mathbf{y}) \geq 0$ with equality if $\mathbf{x} = \mathbf{y}$,
- $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$,
- $\rho(\mathbf{x}, \mathbf{y}) \leq \rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y})$.

is called a metric. Many of the results in this course are actually valid for general metric space.

The third condition is also known as the **triangle inequality**. It states that the distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ is always less than the distance between $\mathbf{x}$ and some third vector $\mathbf{z}$ plus the distance between $\mathbf{z}$ and $\mathbf{y}$. It can be proven from first principles. In order to do this, we first have to prove an interesting intermediate result called the Cauchy-Schwartz inequality.

Theorem 1 (Cauchy-Schwartz and Triangular inequality)**.** *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$

and

$$\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|.$$

Proof. Let us first proof the first inequality, called the Cauchy-Schwartz inequality. Notice that if $\mathbf{y} = \mathbf{0}$, then the inequality holds.

As such, assume that $\mathbf{y} \neq \mathbf{0}$. Let $c \in \mathbb{R}$ and consider the value of $\|\mathbf{x} - c\mathbf{y}\|$. We have,

$$\begin{aligned} 0 &\leq (\|\mathbf{x} - c\mathbf{y}\|)^2, \\ &= \sum_i (x_i - cy_i)^2 = c^2 \sum_i y_i^2 - 2c \sum_i (x_i y_i) + \sum_i x_i^2, \\ &= c^2(\|\mathbf{y}\|)^2 - 2c(\mathbf{x} \cdot \mathbf{y}) + (\|\mathbf{x}\|)^2. \end{aligned}$$

This inequality must hold for all possible values of c. Let us take the value $c = \frac{\mathbf{x} \cdot \mathbf{y}}{(\|\mathbf{y}\|)^2}$, which is well defined as $\mathbf{y} \neq \mathbf{0}$. This gives,

$$\begin{aligned} 0 &\leq \frac{(|\mathbf{x} \cdot \mathbf{y}|)^2}{(\|\mathbf{y}\|)^2} - 2\frac{(|\mathbf{x} \cdot \mathbf{y}|)^2}{(\|\mathbf{y}\|)^2} + (\|\mathbf{x}\|)^2, \\ \leftrightarrow & (\|\mathbf{y}\|)^2 (\|\mathbf{x}\|)^2 \geq (|\mathbf{x} \cdot \mathbf{y}|)^2. \end{aligned}$$

Taking square roots on both sides gives the desired inequality.

Now, for the triangular inequality, consider two vectors $\mathbf{x}$ and $\mathbf{y}$. Then,[27]

$$\begin{aligned} (\|\mathbf{x} + \mathbf{y}\|)^2 = \sum_i (x_i + y_i)^2 &= \sum_i x_i^2 + \sum_i y_i^2 + 2\sum_i (x_i y_i), \\ &\leq \sum_i x_i^2 + \sum_i y_i^2 + 2\left|\sum_i (x_i y_i)\right|, \\ &= (\|\mathbf{x}\|)^2 + (\|\mathbf{y}\|)^2 + 2|\mathbf{x} \cdot \mathbf{y}|, \\ &\leq (\|\mathbf{x}\|)^2 + (\|\mathbf{y}\|)^2 + 2\|\mathbf{y}\| \; \|\mathbf{x}\|, \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \end{aligned}$$

Taking square roots from both sides gives $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|y\|$. Now,[28]

$$\|\mathbf{x} - \mathbf{y}\| = \|(\mathbf{x} - \mathbf{z}) + (\mathbf{z} + \mathbf{y})\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|.$$

□

[27] The fourth line follows from the Cauchy-Schwartz inequality.

[28] Here we treat $(\mathbf{x} - \mathbf{z})$ and $(\mathbf{z} - \mathbf{y})$ as two vectors and apply the previous rule.

As stated above, the inequality,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$

is called the **Cauchy-Schwartz inequality** and is of great interest in its own right. If $\mathbf{x}$ is a scalar, one easily sees that $\|\mathbf{x}\| = |x|$, so the triangular inequality gives as a special case that for all $x, y, z \in \mathbb{R}$,

$$|x - y| \leq |x - z| + |z - y|.$$

Sequences in $\mathbb{R}$

THE STUDY OF sequences and series provides a way to develop intuition abut the notions of arbitrarily large and arbitrarily small numbers. These notions are developed by using the idea of a limit of a sequence of numbers.

A SEQUENCE IS simply a succession of numbers. For example, 1, 4, 9, 16,... could be seen as a sequence of squares of the natural numbers $1^2, 2^2, 3^2, \ldots$. We could easily write down the rule generating this sequence by defining the following function,

$$f(n) = n^2, \qquad n = 1, 2, 3, \ldots$$

Formally, we have the following definition for a sequence in $\mathbb{R}$.

Definition 1 (sequence). *A **sequence** in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$ that associates a real number $f(n)$ with every integer $n \in \mathbb{N}$.*

Instead of writing down a sequence in terms of a function $f(n)$ it is often convenient to use the 'enumeration' notation $(x_n)_{n\in\mathbb{N}}$ where $x_n = f(n)$. This is also the notation that we will use.

Sequences that are getting closer and closer to some particular value as n grows larger are said converge to a limit. Sequences that are getting larger and larger in absolute value as n grows are said to diverge (to infinity). Some sequences neither diverge nor have a limit.[29]

Plot the following sequences on a graph.

- $f(n) = 2n$
- $f(n) = 1/n$
- $f(n) = -1/n$
- $f(n) = -n^2$
- $f(n) = (-2)^n$

Definition 2 (limit). *The sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converges to 0 if for all $\varepsilon > 0$ there is a number N_ε such that for all $n \geq N_\varepsilon$,*[30]

$$|x_n| < \varepsilon.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converges to the number x if the sequence of numbers $(a_n)_{n\in\mathbb{N}}$ with $a_n = |x_n - x|$ converges to 0.

Using the notation of the previous section, we can write this down as,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : |a_n - a| < \varepsilon.$$

In words, a sequence $(x_n)_{n\in\mathbb{N}}$ has a limit x if all values of the sequence, beyond a certain term can be made as close to x as one wishes. If a sequence $(x_n)_{n\in\mathbb{N}}$ has a limit x we write this as

$$\lim_n x_n = x,$$

or

$$x_n \xrightarrow{n} x.$$

As an example, consider the sequence $(x_n)_{n\in\mathbb{N}} = (1/n)_{n\in\mathbb{N}}$. Our best guess of limit for this sequence is $x = 0$. If we take, for example,

[29] Can you say which of the sequences given above diverges or converges.

[30] We use the notation N_ε to make it clear that N_ε may be different for different values of ε.

$\varepsilon = 0.01$ then we have that that $|1/n - 0| < 0.01$ for all $n > 100$, so setting $N_\varepsilon = 100$ will suffice. On the other hand, if we take $\varepsilon = 0.002$, a value of $N_\varepsilon = 500$ will do. In general for a value $\varepsilon > 0$ we will need to take a choice that satisfies $N_\varepsilon > 1/\varepsilon$. Indeed for this choice, we have that if $n > N_\varepsilon > 1/\varepsilon$ then,

$$|x_n - 0| = \left|\frac{1}{n}\right| \leq \frac{1}{N_\varepsilon} < \varepsilon.$$

Let us make this a bit more rigorous and prove that indeed $\frac{1}{n} \xrightarrow{n} 0$. What we need to show is that for all $\varepsilon > 0$ there is a number N_ε such that for all $n \geq N_\varepsilon$,

$$\left|\frac{1}{n} - 0\right| < \varepsilon.$$

This is equivalent to the condition that, for all $n \geq N_\varepsilon$,

$$\frac{1}{n} < \varepsilon.$$

Multiplying both sides by n gives,

$$\begin{aligned} 1 &< n\varepsilon, \\ \leftrightarrow n &> 1/\varepsilon. \end{aligned}$$

Thus choosing N_ε to be the smallest integer above $1/\varepsilon$ will guarantee the validity of the definition.

As a second example, let us show that the sequence $((-1)^n)_{n\in\mathbb{N}}$ does not have a limit. Let us prove this by contradiction. Towards a contradiction, assume that there is an x such that $(-1)^n \xrightarrow{n} x$. In other words, for all $\varepsilon > 0$ there is a number $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$, $|(-1)^n - x| < \varepsilon$.

Let us show that this gives a contradiction for $\varepsilon = 1/2$. Let N be the number such that for all $n > N$

$$|(-1)^n - x| < \varepsilon = \frac{1}{2}.$$

If N is odd, then $N+1$ is even and $N+2$ is odd, so $(-1)^{N+1} = 1$ and $(-1)^{N+2} = -1$ which means that,

$$|1 - x| < 1/2 \text{ and } |-1-x| < 1/2.$$

If N is even then $N+1$ is odd and $N+2$ is even, so $(-1)^{N+1} = -1$ and $(-1)^{N+2} = 1$ which means that,

$$|-1-x| < 1/2 \text{ and } |1-x| < 1/2.$$

In both cases, we see that x should lie within distance $1/2$ of both 1 and -1, which is impossible.[31] This gives the desired contradiction and therefore shows that $(-1)^n$ has no limit.

[31] For x to lie within a distance $1/2$ of 1, we must have that $x \in [0.5, 1.5]$ for x to lie within a distance $1/2$ from -1, we should have that $x \in [-1.5, -0.5]$. Since these two intervals don't overlap, there is no x that lies within a distance $1/2$ from both.

Consider a sequence $x_n \xrightarrow{n} x$ and assume that there is a number $z \in \mathbb{R}$ such that $x_n \leq z$ for all $n \in N$. What do we know about the ranking of x and z. Luckily it does what we expect: $x \leq z$ so, inequalities are preserved in the limit.

Theorem 2. *Consider a sequence $x_n \xrightarrow{n} x$. If for all $n \in \mathbb{N}$, $x_n \leq z$, then $x \leq z$. On the other hand, if for all $n \in \mathbb{N}$, $x_n \geq z$ then $x \geq z$.*

Proof. We only proof the first part. The second part is similar.[32] Let $x_n \xrightarrow{n} x$ and $x_n \leq z$ for all $n \in \mathbb{N}$. We need to show that $x \leq z$. The proof is by contradiction. As such, we assume that

$$z < x.$$

Now, $x_n \xrightarrow{n} x$ so by definition of a limit,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : |x_n - x| < \varepsilon.$$

The idea is to pick a smart choice of ε. Here, we will set $\varepsilon = x - z > 0$. Then, our definition gives the existence of a number $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - x| < \varepsilon = x - z.$$

As such, there exists a number $N \in \mathbb{N}$ such that for all $n \geq N$,

$$x_n > x - \varepsilon = x - x + z = z.$$

This contradicts the requirement that $x_n \leq z$ for all $n \in \mathbb{N}$. □

In a similar vain, one can show that if $x_t \xrightarrow{n} x$ and $z_t \xrightarrow{n} z$ and $z_t \leq x_t$ for all $t \in \mathbb{N}$, then $z \leq x$ (Do this).

Remark: Above theorem also shows that if $x_n \xrightarrow{n} x$ and $x_n < z$ for all $n \in \mathbb{N}$, then $x \leq z$. However, it is **not** true that $x_n < z$ for all $n \in \mathbb{N}$ implies $x < z$. For a counterexample, consider the sequence $x_n = 1 - 1/n$. We have that $x_n < 1$ for all $n \in N$ but $x_n \to 1$ which is not strictly lower than 1.[33]

The following theorem gives some useful rules to work with limits.[34]

Theorem 3. *The following rules apply to all sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.*

1. *If $x_n \xrightarrow{n} x$ and $y_n \xrightarrow{n} y$ then $(x_n + y_n) \xrightarrow{n} (x + y)$,*
2. *If $\alpha \in \mathbb{R}$, then $x_n \xrightarrow{n} x$ implies $(\alpha x_n) \xrightarrow{n} (\alpha x)$.*
3. *If $x_n \xrightarrow{n} x$ and $y_n \xrightarrow{n} y$ then, $(x_n y_n) \xrightarrow{n} (xy)$,*
4. *If $x_n \xrightarrow{n} x$ and $y_n \xrightarrow{n} y$ and $y \neq 0$ then, $\frac{x_n}{y_n} \xrightarrow{n} \frac{x}{y}$.*

[32] Try this.

[33] To summarize: limits of weak inequalities convert to weak inequalities. Limits of strict inequalities do not always convert to strict inequalities.

[34] The proof of these results is simplified with the help of the Convergence Lemma, which is exercise 1. Proving the theorem itself is exercise **??**.

Supremum and infimum

A SUBSET $S \subseteq \mathbb{R}$ has an **upper bound** in $\mathbb{R}$ if there is a number $x \in \mathbb{R}$ such that for all $y \in S$, $x \geq y$. Observe that it is not necessarily the case that an upper bound x of S is also an element of S. Similarly, the set S has a lower bound in $\mathbb{R}$ if there is a number $x \in \mathbb{R}$ such that $x \leq y$ for all $y \in S$.

Exercise: Do the following sets have an upper or lower bound?
$S_1 = \{x \in \mathbb{R} | x \geq 10\}$
$S_2 = \{x \in \mathbb{R}_{+,0} | 1/x < 2\}$
$S_3 = \{x \in \mathbb{R} | x^2 \leq 2\}$

Not every subset $S \subset \mathbb{R}$ has an upperbound. For example $S = \{1, 2, 3, \ldots\}$ is clearly unbounded from above and $S = \{-1, -3, -5, \ldots\}$ is unbounded from below. For sets that are bounded from above (or below) we can define a smallest upperbound (or largest) lower bound.

Definition 3 (Infimum and supremum)**.**

- *If $S \subseteq \mathbb{R}$ is bounded from above, it has a lowest upperbound. This number is called the supremum and we write it as* $\sup S$.

  The formal definition is that $y = \sup S$ iff (i) y is an upper bound of S and (ii) for all other upper bounds z of S, we have that $y \leq z$.

- *If $S \subseteq \mathbb{R}$ is bounded from below, it has a greatest lowerbound. This number is called the infimum and we often write it as* $\inf S$.

  The formal definition is that $y = \inf S$ iff (i) y is a lower bound of S and (ii) for all other lower bounds z of S, we have that $z \leq y$.

Observe that the supremum or infimum of S, if it exist, is not necessary an element of S. The existence of a supremum or infimum for bounded sets follows from the completeness of the real numbers. This is a inherent property of the real numbers that cannot be proven from first principles. In fact, it is a consequence of the way the real numbers are constructed. The infimum and supremum of a set S, if they exist, are unique.[35]

Exercise: What are the infima and suprema of the sets given above?

[35] Indeed, assume that both x and y are suprema of S. Then it must be that $x \leq y$ and $y \leq x$ so we have $x = y$. A similar reasoning holds for the infimum.

The following gives a useful alternative characterization for the supremum and infimum.

Theorem 4. *Let $S \subseteq \mathbb{R}$ be bounded from above. Then $y = \sup S$ if and only if y is an upperbound of S and for all $\varepsilon > 0$ there is an element $x \in S$ such that $y < x + \varepsilon$.*

Let $S \subseteq \mathbb{R}$ be bounded from below. Then $y = \inf S$ if and only if y is a lowerbound of S and for all $\varepsilon > 0$ there is an element $x \in S$ such that $y > x - \varepsilon$.

Proof. We only provide the proof for the supremum.[36] Let $y = \sup S$. Then obviously, y is an upper bound for S. We need to show that,

$$\forall \varepsilon > 0, \exists x \in S : y < x + \varepsilon.$$

[36] The proof for the infimum is similar and left as an exercise.

We show this by contradiction. Negating above formula gives,

$$\exists \varepsilon > 0, \forall x \in S : y \geq x + \varepsilon.$$

Let $\varepsilon > 0$ be a number that satisfies this formula. Then for all $x \in S$,

$$\begin{aligned} & y \geq x + \varepsilon, \\ \leftrightarrow & y - \varepsilon \geq x. \end{aligned}$$

This shows that $y - \varepsilon$ is an upper bound of S.[37] However, y was the supremum of S so $y \leq y - \varepsilon$,[38] which gives a contradiction.

[37] All $x \in S$ are below $y - \varepsilon$ so this is an upper bound of S.

[38] The number y is the smallest upper bound so it is smaller than the upper bound $y - \varepsilon$.

Next, let us show the reverse. Assume that y is an upperbound of S and that,

$$\forall \varepsilon > 0, \exists x \in S : y < x + \varepsilon.$$

We need to show that y is the supremum of S. Again, we prove this by contradiction. Assume that y is not the supremum. We know that y is an upper bound. As y is not the supremum, this means that there is another upperbound of S, say z which is smaller than y, i.e., $z < y$.

Define $\varepsilon > 0$ such that $\varepsilon = y - z > 0$.[39] Then we know that there exists a number $x \in S$ such that,

[39] Here ε is the distance between z and y. Other distances will also work as long as $0 < \varepsilon \leq y - z$. Try this.

$$\begin{aligned} & x + \varepsilon > y = z + \varepsilon, \\ \rightarrow & x > z. \end{aligned}$$

This contradicts the assumption that z is an upper-bound of S.[40] $\square$

[40] Indeed, $x \in S$ and $x > z$ so z is not an upper bound.

As we saw above, not every sequence $(x_n)_{n\in\mathbb{N}}$ has a limit. The following theorem gives an important class of sequences that do have a limit.

Theorem 5. *Any non-decreasing sequence in $\mathbb{R}$ which is bounded from above has a limit in $\mathbb{R}$ and every non-increasing sequence in $\mathbb{R}$ which is bounded from below has a limit in $\mathbb{R}$.*[41]

[41] A sequence $(x_n)_{n\in\mathbb{N}}$ is non-decreasing if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$. A sequence is non-increasing if $x_1 \geq x_2 \geq \ldots \geq x_n \geq \ldots$.

Proof. Let $(x_t)_{t\in\mathbb{N}}$ be a non-decreasing sequence which is bounded from above.[42] Let $y = \sup\{x_n : n \in \mathbb{N}\}$. This supremum exists as S is bounded from above.

[42] This means that the set $S = \{x_n : n \in \mathbb{N}\}$ is bounded from above.

Let us prove that $x_t \to y$. In particular we need to show that,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : |x_n - y| < \varepsilon.$$

Consider any $\varepsilon > 0$. By Theorem 4, we know that there exists an x_N in the sequence such that,

$$y < x_N + \varepsilon.$$

Also, as y is an upper bound of the sequence, we have that,

$$\begin{aligned} &x_N - \varepsilon < y < x_N + \varepsilon, \\ \leftrightarrow &|x_N - y| < \varepsilon. \end{aligned}$$

Now, take any $n \geq N$. As the sequence is non-decreasing,[43] we have that,

[43] In other words, $x_N \leq x_n$.

$$x_n - \varepsilon < y < x_N + \varepsilon \leq x_n + \varepsilon,$$

which means that,

$$|x_n - y| < \varepsilon,$$

for all $n > N$ which we needed to show. The proof for the infimum is similar.[44] □

[44] Try this yourself.

Subsequences

Subsequences are to sequences what subsets are to sets. More formally, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$. Consider an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$,[45]

[45] A function is increasing if $n > m$ implies $f(n) > f(m)$.

$$\varphi(1) < \varphi(2) < \varphi(3) < \dots,$$

Then define the numbers,

$$y_n = x_{\varphi(n)}.$$

The sequence $(y_n)_{n \in \mathbb{N}} = (x_{\varphi(n)})_{n \in \mathbb{N}}$ is also a sequence. It is called a **subsequence** of $(x_n)_{n \in \mathbb{N}}$. The concept is important enough to have its own definition.[46]

[46] Convince yourself of the fact that the two definitions are identical.

Definition 4 (subsequence)**.** *Let $x_n = f(n)$ where $f : \mathbb{N} \to \mathbb{R}$ represents a sequence in $\mathbb{R}$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function,[47] then,*

[47] This means that for $n, m \in \mathbb{N}$ if $n < m$ then $\varphi(n) < \varphi(m)$.

$$y_j = f(\varphi(j)),$$

is called a subsequence of $(x_n)_{n \in \mathbb{N}}$.

We will also denote the subsequence by $(x_{\varphi(n)})_{n \in \mathbb{N}}$, we can also list it by elements,

$$x_{\varphi(1)}, x_{\varphi(2)}, \dots, x_{\varphi(n)}, \dots$$

The following result is immediate given the definition of convergence.

Theorem 6. *Every subsequence of a convergent sequence is also convergent and has the same limit as the original sequence.*

The next result called the Bolzano Weierstrass theorem is less obvious but used over and over again in these notes.

Theorem 7 (Bolzano-Weierstrass)**.** *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is bounded, then it contains a convergent subsequence.*

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is bounded. We call an element x_n from this sequence a **top** if,

$$\forall m \geq n : x_n \geq x_m$$

In other words, x_n is larger or equal to all elements in the sequence beyond x_n. Let T be the set of all tops. There are two cases. Either T is finite or T is infinite.

If T is finite then there is a number N such that for all $n \geq N$, x_n is not a top. Let $\varphi(1) \geq N$ then $x_{\varphi(1)}$ is not a top, so there is a number $n_2 > \varphi(1)$ such that $x_{\varphi(1)} < x_{n_2}$. Let us define $\varphi(2) = n_2$. But $x_{\varphi(2)}$ is also not a top, so there is a number $n_3 > \varphi(2)$ such that $x_{\varphi(2)} < x_{n_3}$. We define $\varphi(3) = n_3$. We can continue this reasoning indefinitely long generating a sequence,

$$x_{\varphi(1)} < x_{\varphi(2)} < x_{\varphi(3)} < x_{\varphi(4)} < \dots$$

Observe that $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. Also this subsequence is non-decreasing and it is bounded from above.[48] As such, from Theorem 5 we see that $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is convergent, which concludes the proof.

[48] As $(x_n)_{n\in\mathbb{N}}$ is bounded from above.

Now what happens if the set of tops T is infinite. Then we can list these tops, say

$$x_{\varphi(1)}, x_{\varphi(2)}, x_{\varphi(3)}, \dots$$

i.e. $\mathbf{x}_{\varphi(n)}$ is equal to the nth top. Also,

$$x_{\varphi(1)} \geq x_{\varphi(2)} \geq x_{\varphi(3)} \geq \dots,$$

so we obtain a non-increasing sequence which is bounded from below. Again from Theorem 5, this sequence has a limit. □

Sequences of vectors

So far, we looked at sequences of real numbers. Fortunately, the concept easily extends to sequences of vectors.

Definition 5 (sequence of vectors)**.** *A **vector sequence** in* $\mathbb{R}^k$ *is a function* $f : \mathbb{N} \to \mathbb{R}^k$ *that associates a real vector* $f(n) = \mathbf{x}_n$ *to each positive integer* $n \in \mathbb{N}$.

The definition of a limit of a sequence of vectors is also analogous to the limit of a sequence of numbers. The only difference is how we measure the distance between elements. For numbers, we used the absolute value $|x - y|$. When dealing with vectors, we use the Euclidean norm $\|\mathbf{x} - \mathbf{y}\|$.[49]

[49] Recall, $\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$.

Definition 6 (limit). *The sequence of vectors $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a limit $\mathbf{x}$ if the sequence of numbers $(a_n)_{n\in\mathbb{N}}$, where $a_n = \|\mathbf{x}_n - \mathbf{x}\|$ converges to 0.*[50]

[50] In other words, for all $\varepsilon > 0$, there is a number $N_\varepsilon \in \mathbb{N}$ such that, for all $n > N_\varepsilon$,
$$\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon.$$

As a formula,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : \|\mathbf{x}_n - \mathbf{x}\| < \varepsilon.$$

The notion of a subsequence of a sequence of vectors is very similar to a subsequence of a sequence of numbers. In particular, we take an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$,

$$\varphi(1) < \varphi(2) < \varphi(3) < \dots$$

Then we define $\mathbf{y}_j = \mathbf{x}_{\varphi(j)}$ to be a subsequence of $(\mathbf{x}_n)_{n\in\mathbb{N}}$. The following result is an immediate generalizations from the one dimensional setting to the k-dimensional setting.

Theorem 8.

If the sequence (of vectors) converges then any subsequence of this sequence also converges to the same limit vector.

Next, we would like to provide an analogue to the Bolzano-Weierstrass theorem but now for sequences of vectors, i.e., every sequence in a bounded set has a convergent subsequence. In order to do this, however, we need a notion of boundedness for sequences of vectors. For sequences of numbers, this was easy. A sequence $(x_n)_{n\in\mathbb{N}}$ was bounded if there exists a number M such that for all $n \in \mathbb{N}$, $|x_n| \leq M$. The definition for sequences of vectors is similar, except that we use the norm $\|.\|$ instead of $|.|$.

Definition 7 (Boundedness). *A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is bounded if there exists a number M such that for all $n \in \mathbb{N}$,*

$$\|\mathbf{x}_n\| \leq M.$$

Above definition states that a sequence is bounded if each of it's elements is within a distance M from the origin.[51]

[51] In other words, the entire sequence lies within a ball of radius M centered at the origin.

Theorem 9 (Bolzano-Weierstrass for vector sequences). *If the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ is bounded, then it contains a convergent subsequence in $\mathbb{R}^k$.*

Proof. We proof the theorem by induction on k. For $k = 1$ the theorem states that if a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is bounded, then it contains a convergent subsequence in $\mathbb{R}$. But this is the Bolzano Weierstrass theorem that we already proved before. So we know this is true.

Assume that the theorem holds up to k and assume that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^{k+1}$. Write each element of the sequence $\mathbf{x}_n$ as,

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{y}_n \\ z_n \end{bmatrix}.$$

Here $\mathbf{y}_n$ is the vector that contains the first k elements of the vector $\mathbf{x}_n$,

$$\mathbf{y}_n = \begin{bmatrix} x_{n,1} \\ x_{n,2} \\ \vdots \\ x_{n,k} \end{bmatrix},$$

and z_n equals the last $k+1$th element of the vector $\mathbf{x}_n$.

Observe that $\|\mathbf{y}_n\| < \|\mathbf{x}_n\|$ and $|z_n| \leq \|\mathbf{x}_n\|$ which shows that both sequences $(\mathbf{y}_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in N}$ are bounded.

We can write the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots$ in the following way,

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{y}_1 \\ z_1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} \mathbf{y}_2 \\ z_2 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} \mathbf{y}_3 \\ z_3 \end{bmatrix}, \ldots$$

By the induction hypothesis, we know that the sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ has a convergent subsequence. Let $(\mathbf{y}_{\varphi(n)})_{n\in\mathbb{N}}$ be this sequence. Let us restrict $(\mathbf{x}_n)_{n\in N}$ to this subsequence,

$$\mathbf{x}_{\varphi(1)} = \begin{bmatrix} \mathbf{y}_{\varphi(1)} \\ z_{\varphi(1)} \end{bmatrix}, \qquad \mathbf{x}_{\varphi(2)} = \begin{bmatrix} \mathbf{y}_{\varphi(2)} \\ z_{\varphi(2)} \end{bmatrix}, \qquad \mathbf{x}_{\varphi(3)} = \begin{bmatrix} \mathbf{y}_{\varphi(3)} \\ z_{\varphi(3)} \end{bmatrix}, \ldots$$

The sequence $(z_{\varphi(n)})_{n\in\mathbb{N}}$ is also a bounded sequence in $\mathbb{R}$, so it has a convergent subsequence. Let us denote this sequence by $(z_{\varphi(\psi(n))})_{n\in\mathbb{N}}$. Let us restrict $(\mathbf{x}_{\varphi(n)})_{n\in N}$ to this subsequence,

$$\mathbf{x}_{\varphi(\psi(1))} = \begin{bmatrix} \mathbf{y}_{\varphi(\psi(1))} \\ z_{\varphi(\psi(1))} \end{bmatrix}, \qquad \mathbf{x}_{\varphi(\psi(2))} = \begin{bmatrix} \mathbf{y}_{\varphi(\psi(2))} \\ z_{\varphi(\psi(2))} \end{bmatrix}, \qquad \mathbf{x}_{\varphi(\psi(3))} = \begin{bmatrix} \mathbf{y}_{\varphi(\psi(3))} \\ z_{\varphi(\psi(3))} \end{bmatrix}, \ldots$$

Observe that $(\mathbf{y}_{\varphi(\psi(n))})_{n\in N}$ is a subsequence of $(\mathbf{y}_{\varphi(n)})_{n\in\mathbb{N}}$, so it also converges. Let $\mathbf{y}_{\varphi(\psi(n))} \xrightarrow{n} \mathbf{y}$ and let $z_{\varphi(\psi(n))} \xrightarrow{n} z$. Define,

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ z \end{bmatrix}.$$

Then

$$\begin{aligned}\|\mathbf{x}_{\varphi(\psi(n))} - \mathbf{x}\| &= \left\| \begin{bmatrix} \mathbf{y}_{\varphi(\psi(n))} \\ z_{\varphi(\psi(n))} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ z_{\varphi(\psi(n))} \end{bmatrix} + \begin{bmatrix} \mathbf{y} \\ z_{\varphi(\psi(n))} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ z \end{bmatrix} \right\|, \\ &\leq \left\| \begin{bmatrix} \mathbf{y}_{\varphi(\psi(n))} \\ z_{\varphi(\psi(n))} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ z_{\varphi(\psi(n))} \end{bmatrix} \right\| + \left\| \begin{bmatrix} \mathbf{y} \\ z_{\varphi(\psi(n))} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ z \end{bmatrix} \right\|, \\ &= \|\mathbf{y}_{\varphi(\psi(n))} - \mathbf{y}\| + |z_{\varphi(\psi(n))} - z|.\end{aligned}$$

The two sequences on the right converge to zero, which means (by the convergence lemma) that $\mathbf{x}_{\varphi(\psi(n))} \overset{n}{\to} \mathbf{x}$. This shows that the subsequence $(\mathbf{x}_{\varphi(\psi(n))})_{n\in\mathbb{N}}$ of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges.

□

Cauchy sequence

Sometimes we would like to know whether a sequence, say $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to some limit vector $\mathbf{x}$ without having to know the limit of the sequence. Towards this end, the concept of a Cauchy sequence is very convenient. To illustrate, consider the following sequence,

$$x_n = f(n) = \sum_{t=1}^{n} \frac{1}{t^2}.$$

This gives the sequence,

$$1,\ 1.25,\ 1.36111,\ 1.423611,\ 1.463611,\ \ldots$$

Does this sequence converge? Alternative, consider the sequence,

$$x_n = g(n) = \sum_{t=1}^{n} \frac{1}{t}.$$

This produces the numbers,

$$1,\ 1.5,\ 1.8333,\ 2.0833,\ 2.2833,\ \ldots$$

Does this sequence converge? In order to answer these questions, the straightforward thing is to go back to the definition of convergence, we have that $(x_t)_{t\in\mathbb{N}}$ converges if there is an x such that,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : |x_n - x| < \varepsilon.$$

In order to verify this definition it is necessary to know the value of the limiting value x. In some settings, it is possible to make an educated guess. In other settings[52] making such guess is not really straightforward. If so, the notion of a Cauchy sequence is very convenient.

[52] Like the ones above.

Definition 8 (Cauchy sequence)**.** *A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is a number $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$, $\|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon$.*

Written down as a formula, we have that,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n, m \geq N_\varepsilon : \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon.$$

The idea behind a Cauchy sequence is that elements arbitrarily far in the sequence eventually become arbitrarily close together. Notice that this definition does not involve a limiting value $\mathbf{x}$. So verifying whether a sequence is a Cauchy sequence only depends on the values in the sequence itself.

Remark Observe that this definition is not the same as

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : \|x_n - x_{n+1}\| < \varepsilon,$$

i.e. subsequent terms become arbitrarily close together. Any Cauchy sequence satisfies this second condition but not every sequence that satisfies this second condition is a Cauchy sequence.[53]

[53] We will see a counterexample below.

Convergent sequences in $\mathbb{R}^k$ turn out to be Cauchy sequences and Cauchy sequences are the sequences that converge. This is the main message of the following theorem.

Theorem 10. *Any converging sequence is a Cauchy sequence and every Cauchy sequence converges.*

Proof. In order to see the first, assume $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$. Then, by definition, for all $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, $\|\mathbf{x} - \mathbf{x}_n\| < \varepsilon/2$.

Then, for all $n, m \geq N_\varepsilon$,[54]

[54] The second line uses the triangle inequality.

$$\begin{aligned}
\|\mathbf{x}_n - \mathbf{x}_m\| &= \|\mathbf{x}_n - \mathbf{x} + \mathbf{x} - \mathbf{x}_m\|, \\
&\leq \|\mathbf{x} - \mathbf{x}_n\| + \|\mathbf{x}_m - \mathbf{x}\|, \\
&< 2\frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

This shows that the sequence is Cauchy.

The reverse, that any Cauchy sequence is convergent, uses the Bolzano-Weierstrass theorem. Assume that $(\mathbf{x}_t)_{t\in\mathbb{N}}$ is a Cauchy sequence. In order to use the Bolzano-Weierstrass theorem, we first need to show that this sequence is bounded. We know that

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall n, m \geq N_\varepsilon : \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon.$$

Take $\varepsilon = 1$.[55] Then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$,

[55] Other values are also possible.

$$\|\mathbf{x}_n - \mathbf{x}_m\| < 1.$$

Let $M = 1 + \max\{\|\mathbf{x}_r\| : r \leq N\}$. We will show that for all $n \in \mathbb{N}$, $\|\mathbf{x}_n\| \leq M$.

Take any $\mathbf{x}_n$ in the sequence then either $n < N$, but then $\|\mathbf{x}_n\| < M$ so $\|\mathbf{x}_n\|$ is bounded by M. If $n \geq N$ then,

$$\begin{aligned} \|\mathbf{x}_n\| &= \|\mathbf{x}_n - \mathbf{x}_N + \mathbf{x}_N\| \leq \|\mathbf{x}_n - \mathbf{x}_N\| + \|\mathbf{x}_N\| < 1 + \|\mathbf{x}_N\|, \\ \rightarrow &\|\mathbf{x}_n\| < \|\mathbf{x}_N\| + 1 \leq M. \end{aligned}$$

So again $\|\mathbf{x}_m\|$ is bounded by M as was to be shown.

Knowing that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is bounded, we can apply the Bolzano-Weierstrass theorem on the Cauchy sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$, so there is a convergent subsequence $(\mathbf{x}_{\varphi(n)})_{n\in\mathbb{N}}$. Let $\mathbf{x}_{\varphi(n)} \overset{n}{\to} \mathbf{x}$. We will finish the proof by showing that $\mathbf{x}$ is also the limit of the Cauchy sequence $(\mathbf{x}_n)_{n\in N}$.

In particular, we need to show that,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, n \geq N_\varepsilon : \|\mathbf{x}_n - \mathbf{x}\| < \varepsilon.$$

Take any number $\varepsilon > 0$ then as for the subsequence $\mathbf{x}_{\varphi(n)} \overset{n}{\to} \mathbf{x}$, there is a number N_1 such that for all $\varphi(n) \geq N_1$ in the subsequence,

$$\|\mathbf{x} - \mathbf{x}_{\varphi(n)}\| < \frac{\varepsilon}{2}.$$

Moreover as $(\mathbf{x}_n)_{n\in N}$ is Cauchy, there is a number N_2 such that for all $n, m \geq N_2$,

$$\|\mathbf{x}_n - \mathbf{x}_m\| < \frac{\varepsilon}{2}.$$

Now, take a vector $\mathbf{x}_{\varphi(k)}$ in the convergent subsequence with

$$\varphi(k) \geq N_\varepsilon = \max\{N_1, N_2\}.$$

Then for all $n \geq N_\varepsilon$,

$$\begin{aligned} \|\mathbf{x}_n - \mathbf{x}\| &= \|\mathbf{x}_n - \mathbf{x}_{\varphi(k)} + \mathbf{x}_{\varphi(k)} - \mathbf{x}\|, \\ &\leq \|\mathbf{x}_n - \mathbf{x}_{\varphi(k)}\| + \|\mathbf{x}_{\varphi(k)} - \mathbf{x}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\mathbf{x}_n \overset{n}{\to} \mathbf{x}$.[56] □

[56] The idea is of this construction is the following: we take a vector in the subsequence that is far enough such that (i) its distance from **x** is smaller than $\varepsilon/2$ and (ii) its distance from any other vector further in the sequence is also smaller than $\varepsilon/2$. If (i) and (ii) are satisfied, then the distance between any of these vectors further on in the sequence and **x** will be smaller than ε.

Given our machinery of Cauchy sequences let us return to the two examples at the beginning of this section. We had the sequence,

$$x_n = f(n) = \sum_{t=1}^{n} \frac{1}{t^2}.$$

This sequence is convergent if and only if it is a Cauchy sequence, i.e.,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n, m \geq N_\varepsilon : |x_n - x_m| < \varepsilon.$$

Now, take n, m and assume without loss of generality that $m > n$ then,

$$
\begin{aligned}
|x_n - x_m| &= \left| \sum_{t=1}^{n} \frac{1}{t^2} - \sum_{t=1}^{m} \frac{1}{t^2} \right|, \\
&= \left| \sum_{t=n+1}^{m} \frac{1}{t^2} \right|, \\
&= \sum_{t=n+1}^{m} \frac{1}{t^2} \leq \sum_{t=n+1}^{m} \frac{1}{t(t-1)}, \\
&= \sum_{t=n+1}^{m} \left(\frac{1}{t-1} - \frac{1}{t} \right), \\
&= \frac{1}{n} - \frac{1}{m} \leq \frac{1}{n}.
\end{aligned}
$$

So we only need to take N_ε such that

$$
\begin{aligned}
\frac{1}{N_\varepsilon} &< \varepsilon, \\
\to N_\varepsilon &> \frac{1}{\varepsilon}.
\end{aligned}
$$

As such, we see that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so it converges.[57] As an alternative proof of convergence, notice that the series, $(x_n)_{n\in\mathbb{N}}$ are non-decreasing. As such, using Theorem 5, it suffices to show that the sequence is bounded from above.

[57] In fact, it can be shown that

$$\sum_{t=1}^{n} \frac{1}{t^2} = x_n \xrightarrow{n} \frac{\pi^2}{6}.$$

This is the so called Basel problem which was solved by Euler in 1734 at the age of 28. A proof of this result is well beyond the scope of these notes.

Towards this end, let us show that $|x_n| \leq 2$. Indeed, this is true for $x_1 = 1$. For $n \geq 1$, we have,

$$
\begin{aligned}
x_n = \sum_{t=1}^{n} \frac{1}{t^2} &\leq 1 + \sum_{t=2}^{n} \frac{1}{t(t-1)}, \\
&= 1 + \sum_{t=2}^{n} \left(\frac{1}{(t-1)} - \frac{1}{t} \right), \\
&= 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n} \leq 2.
\end{aligned}
$$

So $|x_n| \leq 2$ for all $n \in \mathbb{N}$ which shows that $(x_n)_{n\in\mathbb{N}}$ is bounded.

Next, let us consider the second sequence defined above,

$$x_n = f(n) = \sum_{k=1}^{n} \frac{1}{k}.$$

It turns out that this sequence is not convergent. One way to prove this is to show that it is not a Cauchy sequence. In particular, we can show that,

$$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n, m \geq N : |x_n - x_m| \geq \varepsilon.$$

Towards this end, take a number $N \in \mathbb{N}$ and two numbers $n, m \geq N$ with $m > n$ we have,

$$
\begin{aligned}
|x_m - x_n| &= \left| \sum_{t=n}^{m} \frac{1}{t} \right|, \\
&= \sum_{t=n}^{m} \frac{1}{t} \geq \frac{1}{m}(m-n) = 1 - \frac{n}{m}.
\end{aligned}
$$

Now, we are free to choose m as long as $m \geq n$. So take $m = 2n$ then,[58]

$$|x_m - x_n| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

As such, we see that for all $N \in \mathbb{N}$ we can find $n, m \geq N$ such that $|x_n - x_m| \geq \frac{1}{2}$ which shows that $(x_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence.[59]

The sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is called the **harmonic series**.[60]

The following gives an alternative characterization of convergence that will be useful later on.

Lemma 1. *Let $(\mathbf{x}_n)_{n \in \mathbb{N}}$ be a sequence. This sequence converges to a vector $\mathbf{x}$ if and only if every subsequence $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$ of $(\mathbf{x}_n)_{n \in \mathbb{N}}$ has a further subsequence that converges to $\mathbf{x}$.*

Proof. ($\rightarrow$) easy. ($\leftarrow$) The proof is by contrapositive. Assume that $(\mathbf{x}_n)_{n \in \mathbb{N}}$ does not converge to $\mathbf{x}$. Then by definition, there is a $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n \geq N$ such that,

$$\|\mathbf{x}_n - \mathbf{x}\| \geq \varepsilon.$$

From this, we will construct a sequence $(\mathbf{x}_\varphi(n))_{n \in \mathbb{N}}$. Let

$$
\begin{aligned}
&N = 1 \rightarrow \exists n \geq N : \|\mathbf{x}_n - \mathbf{x}\| \geq \varepsilon, \text{ set } \varphi(1) = n, \\
&N = \varphi(1) + 1 \rightarrow \exists n > \varphi(1) : \|\mathbf{x}_n - \mathbf{x}\| \geq \varepsilon, \text{ set } \varphi(2) = n, \\
&N = \varphi(2) + 1 \rightarrow \exists n > \varphi(2) : \|\mathbf{x}_n - \mathbf{x}\| \geq \varepsilon, \text{ set } \varphi(3) = n, \\
&\ldots \\
&N = \varphi(t) + 1 \rightarrow \exists n > \varphi(t) : \|\mathbf{x}_n - \mathbf{x}\| \geq \varepsilon, \text{ set } \varphi(t+1) = n, \\
&\ldots
\end{aligned}
$$

Notice that $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$ is a subsequence of $(\mathbf{x}_n)_{n \in \mathbb{N}}$ and that for any $\mathbf{x}_{\varphi(n)}$ in this sequence,

$$\|\mathbf{x}_{\varphi(n)} - \mathbf{x}\| \geq \varepsilon.$$

Given this, $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$ has no subsequence that converges to $\mathbf{x}$ (as every element is ε far away from $\mathbf{x}$). □

Closed and open sets

[58] Other multiples of n are also possible. Try this.

[59] By setting $\varepsilon = 1/2$.

[60] The fact that the harmonic series does not converge was first given by Oresme in the 14th century. The non-convergence of the harmonic sum leads to some counterintuitive results. One example is the leaning tower of Lire. This shows that you can stack a sequence of blocks on the edge of a table such that they do not fall to the ground and the overhang is arbitrary large provided, of course, that you have a (really) large number of blocks.

A SET IS closed if it contains all limit points of converging sequences in the set. In particular the set S is closed if for all sequences $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$ and if $\mathbf{x}_n \in S$ for all $n \in \mathbb{N}$ then $\mathbf{x} \in S$.

Definition 9 (Closed sets). *A set $S \subseteq \mathbb{R}^k$ is **closed** if for any sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in S*[61] *that has a limit $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$, we also have that this limit is in S, i.e. $\mathbf{x} \in S$.*

[61] In other words, $\mathbf{x}_n \in S$ for all $n \in \mathbb{N}$

A set is open if any element in the set is the center of a small ball that is entirely contained within the set.

Definition 10 (Open set). *A set $S \subseteq \mathbb{R}^k$ is **open** if for all $\mathbf{x} \in S$ there is a $\varepsilon > 0$ such that for all $\mathbf{y} \in \mathbb{R}^k$ with $\|\mathbf{y} - \mathbf{x}\| < \varepsilon$, $\mathbf{y} \in S$.*

Above definition can be summarized using the following formula.

$$\forall \mathbf{x} \in S, \exists \varepsilon > 0, \forall \mathbf{y} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}\| < \varepsilon \to \mathbf{y} \in S.$$

There are sets that are neither open nor closed.[62] For example the half open interval $S =]a, a+1]$ is neither open nor closed.

To see that it is not closed. Take the sequence $x_n = a + 1/n$. We have that $x_n \xrightarrow{n} a$ but $a \notin S$ so S is not closed. In order to show that S is not open, it suffices to consider the point $a+1$. Now for any $\varepsilon > 0$ there are numbers in $]a+1-\varepsilon, a+1+\varepsilon[$ that are not in $]a, a+1]$, so S is not open. Given this counterexample, we see that it is **not** true that a set is open if it is not closed.

[62] In fact there are sets that are both open and closed. These are called clopen. For $\mathbb{R}^k$, we have that $\mathbb{R}^k$ and $\emptyset$ are the only two clopen sets. There is a nice proof by contradiction about this along the following lines. Let A be a clopen set which is neither $\mathbb{R}^k$ nor $\emptyset$. Then there a vectors $\mathbf{y} \in A$ and $\mathbf{z} \in A^c$. For $\theta \in [0,1]$, let $\mathbf{w}(\theta) = \theta\mathbf{y} + (1-\theta)\mathbf{z}$. Then $\mathbf{w}(0) = \mathbf{z} \notin A$ and $\mathbf{w}(1) = \mathbf{y} \in A$. Let $\mathbf{w}^* = \arg\inf_\theta \mathbf{w}(\theta)$ s.t. $\mathbf{w}(\theta) \in A$. Then show that $\mathbf{w}^*$ is both in and outside A.

The following, however is true.

Theorem 11. *If S is open then its complement $\mathbb{R}^k \setminus S \equiv S^c$ is closed. On the other hand, if C is closed then $\mathbb{R}^k \setminus C \equiv C^c$ is open.*

Proof. Let S be open.

$$\forall \mathbf{x} \in S, \exists \varepsilon > 0, \forall \mathbf{y} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}\| < \varepsilon \to \mathbf{y} \in S.$$

We show that $S^c = \mathbb{R}^k \setminus S$ is closed by contradiction. Assume that S^c is not closed. Then there is a convergent sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in S^c (say $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$) with $\mathbf{x} \notin S^c$.

Given that $\mathbf{x} \notin S^c$ it must be that $\mathbf{x} \in S$ as S is the complement of S^c. As S is open, there is a ε such that $\mathbf{y} \in S$ for all $\mathbf{y}$ that satisfies $\|\mathbf{y} - \mathbf{x}\| < \varepsilon$. Also as $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$ we have that there exists a N_ε such that for all $n \geq N$,

$$\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon.$$

This means that for all $n \geq N_\varepsilon$, $\mathbf{x}_n \in S$. This contradicts the assumption that $(\mathbf{x}_n)_{n\in N}$ was a sequence in $S^c = \mathbb{R}^k \setminus S$.

Next let C be closed. We need to show that $C^c = \mathbb{R}^k \setminus C$ is open. Again, we prove this by contradiction. If C^c is not open then[63]

$$\exists \mathbf{x} \in C^c, \forall \varepsilon > 0, \exists \mathbf{y}_\varepsilon \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}_\varepsilon\| < \varepsilon \wedge \mathbf{y}_\varepsilon \notin C^c.$$

[63] Here we use the notation $\mathbf{y}_\varepsilon$ to make clear that $\mathbf{y}$ may change according to the value of ε.

Take this $\mathbf{x} \in C^c$ and consider the following sequence of values of ε and the following vectors $\mathbf{z}_n$.

$$
\begin{aligned}
&\varepsilon = 1 \to \exists \mathbf{y}_\varepsilon : \|\mathbf{x} - \mathbf{y}_\varepsilon\| < 1 \text{ set } \mathbf{z}_1 = \mathbf{y}_\varepsilon \notin C^c, \\
&\varepsilon = \frac{1}{2} \to \exists \mathbf{y}_\varepsilon : \|\mathbf{x} - \mathbf{y}_\varepsilon\| < \frac{1}{2} \text{ set } \mathbf{z}_2 = \mathbf{y}_\varepsilon \notin C^c, \\
&\varepsilon = \frac{1}{3} \to \exists \mathbf{y}_\varepsilon : \|\mathbf{x} - \mathbf{y}_\varepsilon\| < \frac{1}{3} \text{ set } \mathbf{z}_3 = \mathbf{y}_\varepsilon \notin C^c \\
&\vdots \\
&\varepsilon = \frac{1}{n} \to \exists \mathbf{y}_\varepsilon : \|\mathbf{x} - \mathbf{y}_\varepsilon\| < \frac{1}{n} \text{ set } \mathbf{z}_n = \mathbf{y}_\varepsilon \notin C^c \\
&\vdots
\end{aligned}
$$

Consider the sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$. Then,

$$\|\mathbf{z}_n - \mathbf{x}\| < \frac{1}{n}.$$

Given that $(1/n) \xrightarrow{n} 0$, the convergence lemma tells us that $\mathbf{z}_n \xrightarrow{n} \mathbf{x}$. Also, for all $n \in \mathbb{N}$, $\mathbf{z}_n \in C$. As C is closed and $\mathbf{z}_n \xrightarrow{n} \mathbf{x}$ it must be that $\mathbf{x} \in C$. This means that $\mathbf{x} \notin C^c$, a contradiction. □

Closed and bounded sets are called compact.

Definition 11 (Compact sets). *A set $C \subseteq \mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Compact sets have two desirable properties that are often convenient to work with. First, they are bounded, so every sequence in a compact set $C \subseteq \mathbb{R}^k$ has a convergent subsequence, by the Bolzano-Weierstrass theorem. Second, they are closed, which means that the limit of this convergent subsequence is also in C.

Corollary 1. *If $C \subseteq \mathbb{R}^k$ is compact then every sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in C has a convergent subsequence and the limit of this subsequence is also in C.*

If C is a subset of $\mathbb{R}$ (i.e. it contains real numbers, not vectors) and if C is compact, then it contains both its supremum and infimum.

Theorem 12. *If $S \subseteq \mathbb{R}$ and S is compact then* $\sup S \in S$ *and* $\inf S \in S$.

Proof. We prove the theorem for the supremum. The proof for the infimum is similar.[64] Let S be a subset of $\mathbb{R}$. As S is compact, it is bounded, so $\sup S$ and $\inf S$ exist. Let $y = \sup S$. By Theorem 4 we know that,

$$\forall \varepsilon > 0, \exists x_\varepsilon \in S : y \leq x_\varepsilon + \varepsilon.$$

[64] Try it.

We construct the following sequence in S,

$$\begin{aligned}
&\varepsilon = 1 \to \text{ take } x_1 \in S \text{ such that } y \le x_1 + 1,\\
&\varepsilon = \frac{1}{2} \to \text{ take } x_2 \in S \text{ such that } y \le x_2 + \frac{1}{2},\\
&\varepsilon = \frac{1}{3} \to \text{ take } x_3 \in S \text{ such that } y \le x_3 + \frac{1}{3},\\
&\dots,\\
&\varepsilon = \frac{1}{n} \to \text{ take } x_n \in S \text{ such that } y \le x_n + \frac{1}{n},\\
&\dots
\end{aligned}$$

So for all n.

$$|x_n - y| \le \frac{1}{n}.$$

the right hand side $(1/n)$ converges to zero. As such, by the convergence lemma, $x_n \xrightarrow{n} y$. As S is closed and $(x_n)_{n\in N}$ is a sequence in S, we conclude that $y \in S$. □

Exercises

1. Proof the following:

   Lemma 2 (Convergence lemma). *Let $(y_n^1)_{n\in\mathbb{N}}, \dots, (y_n^R)_{n\in\mathbb{N}}$ be R sequences in $\mathbb{R}$ and assume that for all $j = 1, \dots R$, $y_n^j \xrightarrow{n} 0$.*
   Let also $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and assume that there exist numbers $\alpha^j \neq 0$ ($j = 1, \dots, R$) and a number $N \in \mathbb{N}$ such that for all $n \ge N$,

   $$|x_n| \le \sum_{j=1}^{R} \alpha^j |y_n^j|.$$

   Then $x_n \xrightarrow{n} 0$.

2. For each of the following sets S find $\sup S$ and $\inf S$ if they exist.

   - $S = \{x \in \mathbb{R} | x^2 < 5\}$.
   - $S = \{x \in \mathbb{R} | x^2 > 7\}$.
   - $S = \{-1/n | n \in \mathbb{N}\}$.

3. Let $A \subseteq \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ such that $|f(x)| \le M_1$ and $|g(x)| \le M_2$ for all $x \in A$. Show the following

   - $\sup\{f(x) + g(x) | x \in A\} \le \sup\{f(x) | x \in A\} + \sup\{g(x) | x \in A\}$.

   - $\inf\{f(x) + g(x) | x \in A\} \ge \inf\{f(x) | x \in A\} + \inf\{g(x) | x \in A\}$.

- $\sup\{-f(x)|x \in A\} = -\inf\{f(x)|x \in A\}$.
- $\sup\{f(x) - g(x)|x \in A\} \leq \sup\{f(x)|x \in A\} - \inf\{g(x)|x \in A\}$.

4. Show that if $(x_n)_{n\in\mathbb{N}}$ is Cauchy, then $((x_n)^2)_{n\in\mathbb{N}}$ is also Cauchy. Show that the reverse is not true, i.e., provide an example of a sequence such that $((x_n)^2)_{n\in\mathbb{N}}$ is Cauchy but $(x_n)_{n\in\mathbb{N}}$ is not.

5. Let $(x_n)_{t\in\mathbb{N}}$ be a Cauchy sequence such that x_n is an integer for all $n \in \mathbb{N}$. Show that there is a positive integer N such that $x_n = C$ for all $n \geq N$ where C is a constant.

6. Consider any two points $\mathbf{x}^1$ and $\mathbf{x}^2$ in $\mathbb{R}^n$ with $\mathbf{x}^1 \neq \mathbf{x}^2$. Let $B(\mathbf{x}^1, \varepsilon)$ be any open ball centred around $\mathbf{x}^1$. Let
$$Z^* = \{\mathbf{z}|\mathbf{z} = t\mathbf{x}^1 + (1-t)\mathbf{x}^2, t \in]0,1[\}.$$
Prove that $B(\mathbf{x}^1, \varepsilon) \cap Z^* \neq \emptyset$.

7. Consider intervals in $\mathbb{R}$ of the form $[a, +\infty[$ and $]-\infty, b]$. Prove that they are both closed sets. Is the same true for intervals of the form $[a, c[$ and $]-c, b]$ for c finite?

8. Let $S \subseteq \mathbb{R}$ be a set consisting of a single point, $S = \{s\}$. Prove that S is a closed set.

9. Let $\mathbf{x} \in \mathbb{R}^k$. Show that the following set is open:
$$B(x, \varepsilon) = \{\mathbf{y} \in \mathbb{R}^k | \|y - \mathbf{x}\| < \varepsilon\}$$

Functions

FUNCTIONS ARE ONE of the most important concepts in mathematics. Restricting ourselves to the current setting we define a real valued function $f : S \subseteq \mathbb{R}^k \to \mathbb{R}$ as a mapping that associates with every vector $\mathbf{x}$ in a set $S \subseteq \mathbb{R}^k$ a number $f(\mathbf{x}) \in \mathbb{R}$. A multivariate function $f : S \subseteq \mathbb{R}^k \to \mathbb{R}^\ell$ associates to every vector $\mathbf{x} \in S$ another vector $\mathbf{y} \in \mathbb{R}^\ell$.

In this section, we will focus on real valued functions. Multivariate functions will be encountered later on. Functions have the defining property that if $\mathbf{x} = \mathbf{y}$ then $f(\mathbf{x}) = f(\mathbf{y})$. Identical vectors map to identical numbers. The following gives a rather abstract, but mathematically correct definition of a real valued function.

Definition 12 (Function). *A real valued function f with domain $S \subseteq \mathbb{R}^k$ can be defined by a subset $G \subseteq \mathbb{R}^k \times \mathbb{R}$. Such that,*

- *For all $\mathbf{y} \in S$, there is a number $x \in \mathbb{R}$ such that $(\mathbf{y}, x) \in G$,*
- *If $(\mathbf{y}, x) \in G$ and $(\mathbf{y}, z) \in G$ then $x = z$.*

If $(\mathbf{y}, x) \in G$ we call x the function value of $\mathbf{y}$ and write $x = f(\mathbf{y})$.

The **range** of a function $f : S \to \mathbb{R}$ is the set,

$$\{x \in \mathbb{R} : \exists \mathbf{y} \in S, x = f(\mathbf{y})\}.$$

The range of f is often denoted by $f(S)$.[65]

[65] Although S is a subset of $\mathbb{R}^k$, the range $f(S)$ is a subset of $\mathbb{R}$.

A FUNCTION $f : S \to \mathbb{R}$ is continuous at the point $\mathbf{x} \in S$ if a small change in $\mathbf{x}$ causes a small change in the value of $f(\mathbf{x})$. The usual way to define continuity is via the use of the ε-δ formula.

$$\forall \varepsilon > 0, \exists \delta > 0, \forall \mathbf{y} \in S : \|\mathbf{x} - \mathbf{y}\| < \delta \to |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon.$$

Instead of this definition, we will use the following one.[66]

[66] **Exercise:** Show that the two definitions are in fact the same.

Definition 13. *A function $f : S \to \mathbb{R}$ is continuous at $\mathbf{x} \in S$ if for all sequences $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in S,*

$$\textit{if } \mathbf{x}_n \xrightarrow{n} \mathbf{x}, \textit{ then } f(\mathbf{x}_n) \xrightarrow{n} f(\mathbf{x}).$$

Observe that $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$ deals with the convergence of a sequence of vectors. On the other hand $f(\mathbf{x}_n) \xrightarrow{n} f(\mathbf{x})$ is about the convergence of a sequence of real numbers, the sequence $(y_n)_{n\in\mathbb{N}}$ where $y_n = f(\mathbf{x}_n)$ for all $n \in \mathbb{N}$.

Definition 14. *A function f on a domain S is continuous if it is continuous at every vector $\mathbf{x} \in S$.*

Intuitively, a function f is continuous if you can draw it without lifting your pen.

The following lemma provides an alternative characterization of continuity, which we will use later on.

Lemma 3. *Let $f : D \to \mathbb{R}$ be a real valued function with $D \subseteq \mathbb{R}^k$. Then the function f is continuous at $\mathbf{x} \in D$ if and only if every sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in D with $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$ has a subsequence $(\mathbf{x}_{\varphi(n)})_{n\in\mathbb{N}}$ such that $f(\mathbf{x}_{\varphi(n)}) \xrightarrow{n} f(\mathbf{x})$.*

Proof. ($\rightarrow$) Let f be continuous at $\mathbf{x}$ and let $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$. Then for any subsequence $(\mathbf{x}_{\varphi(n)})_{n\in\mathbb{N}}$ we have that $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$. So by continuity $f(\mathbf{x}_{\varphi(n)}) \xrightarrow{n} f(\mathbf{x})$.

($\leftarrow$) Now for the reverse, we prove it by contrapositive. Assume that f is not continuous at $\mathbf{x}$. We want to show that there is a sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$ in D with $\mathbf{z}_n \xrightarrow{n} \mathbf{x}$ such that for all subsequences $(\mathbf{z}_{\varphi(n)})_{n\in\mathbb{N}}$ we have that $f(\mathbf{z}_{\varphi(n)}) \stackrel{n}{\not\to} f(\mathbf{x})$.

As f is not continuous at $\mathbf{x}$, there is a sequence $\mathbf{x}_n \to \mathbf{x}$ and $f(\mathbf{x}_n) \stackrel{n}{\not\to} f(\mathbf{x})$. But then, by Lemma 1, $(f(\mathbf{x}_n))_{n\in\mathbb{N}}$ must have a subsequence, say $(f(\mathbf{x}_{\psi(n)}))_{n\in\mathbb{N}}$ that has no further subsequence that converges to $f(\mathbf{x})$.

However, $(\mathbf{x}_{\psi(n)})_{n\in\mathbb{N}}$ is itself a sequence that converges to $\mathbf{x}$, i.e. $\mathbf{x}_{\psi(n)} \xrightarrow{n} \mathbf{x}$. So there is indeed a sequence in D that converges to $\mathbf{x}$ (namely $(\mathbf{z}_n)_{n\in\mathbb{N}} = (\mathbf{x}_{\psi(n)})_{n\in\mathbb{N}}$) such that for all subsequences, (e.g. $(\mathbf{z}_{\varphi(n))})_{n\in\mathbb{N}} = (\mathbf{x}_{\varphi(\psi(n)})_{n\in\mathbb{N}}$), we have:

$$f(\mathbf{z}_{\varphi(n)}) = f(\mathbf{x}_{\varphi(\psi(n))}) \stackrel{n}{\not\to} f(x).$$

$\square$

Exercises

1. Let A and B be two sets in the domain of f with $B \subseteq A$. Prove that $f(B) \subseteq f(A)$.

2. Let A and B be two sets in the range of f with $B \subseteq A$. Prove that $f^{-1}(B) \subseteq f^{-1}(A)$.

3. For any mapping $f : D \to R$ and any collection of sets A_i in the range of f, show that

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i),$$

and

$$f^{-1}\left(\bigcap_i A_i\right) = \bigcap_i f^{-1}(A_i).$$

4. Let S and T be two nonempty subsets of $\mathbb{R}$, and take any two functions $f : T \to S$ and $g : S \to \mathbb{R}$. Show that if f is continuous at $x \in T$ and g is continuous at $f(x)$. then $g \circ f$ is continuous at x.

Extreme and intermediate value theorem

In this chapter we are going to present and proof two important theorems. The extreme value theorem and the intermediate value theorem.

Extreme value theorem

Let $C \subseteq \mathbb{R}^k$ and let $f : C \to \mathbb{R}$ be a real valued function. Consider the following problem.

$$\sup_{\mathbf{x} \in C} f(\mathbf{x}).$$

If f is unbounded on C, then this problem has no solution.[67] However, if f is bounded from above on C, then the sup is well defined and returns a finite solution.

[67] Or we could say that the solution is equal to ∞.

Example:
Let $C = [0, 1)$ and let $f(x) = x$. It is easy to see that,

$$1 = \sup_{x \in C} f(x).$$

However, there is no value $x \in C$ such that $f(x) = 1$. So the supremum is not attained in C.

Let $C = [0, \infty)$ and let $f(x) = x$. Here $\sup_{x \in C} f(x)$ does not exist, as the range $f(C)$ is unbounded from above.

Let $C = [-1, 1]$ and let $f(x) = 0$ for $x \leq 0$ and $f(x) = 1 - x$ for $x > 0$. Then again,

$$1 = \sup_{x \in C} f(x).$$

but again the supremum is not attained in C.

Figure 2: The function $f(x) = x$ has a supremum but no maximum on $[0, 1)$.

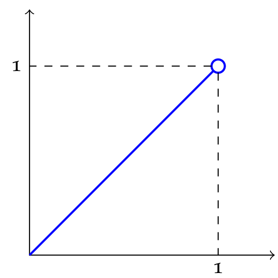
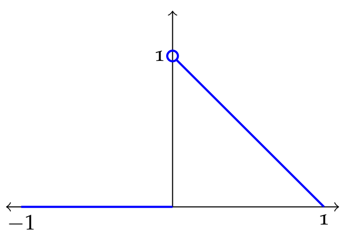


Figure 3: The function $f(x) = 0$ for $x \leq 0$ and $f(x) = 1 - x$ for $x > 0$ has a supremum but no maximum on $[-1, 1]$



If the supremum exist and is actually attained in the set C,[68] we call it a maximum of the function f on C.

[68] In other words, there is an $\mathbf{x} \in C$ such that $f(\mathbf{x}) = \sup_{\mathbf{y} \in C} f(\mathbf{y})$.

Definition 15. *Let $C \subseteq \mathbb{R}^k$ and $f : C \to \mathbb{R}$ a real valued function with domain C. Then $\mathbf{x}_0 \in C$ is a maximum of f on C if*

$$\forall \mathbf{x} \in C : f(\mathbf{x}_0) \geq f(\mathbf{x}).$$

Also $\mathbf{x}_1$ *is a minimum of* f *on* C *if*

$$\forall \mathbf{x} \in C : f(\mathbf{x}_1) \leq f(\mathbf{x}).$$

The three counterexamples given above shows the intricacies. For the first counterexample, the basic problem was that the domain $C = [0, 1)$ did not include the point 1. For the second counterexample, the problem was that C was unbounded (from above). For the third example, the function f is discontinuous at the origin. The extreme value theorem shows that if these deficiencies are left out, i.e. the function f is continuous and the domain C is compact, then not only does the supremum exist but it is also equal to the maximum, i.e.,

$$\sup_{\mathbf{x} \in C} f(\mathbf{x}) = \max_{\mathbf{x} \in C} f(\mathbf{x}).$$

Before we give the proof we first show an intermediate result, namely that the range of a continuous function on a compact set is also compact.

Theorem 13. *If* f *is continuous on a domain* C *and* C *is compact, then* $f(C)$ *is also compact.*

Proof. Let f be continuous and let C be compact. We need to show that $f(C) \subseteq \mathbb{R}$ is also compact. In particular, we need to show that $f(C)$ is bounded and closed.

Let us first show that $f(C)$ is bounded. The proof is by contradiction. If $f(C)$ is not bounded, then[69]

$$\forall M \in \mathbb{N}, \exists \mathbf{x}_M \in C : |f(\mathbf{x}_M)| \geq M.$$

We construct a sequence of vectors $(x_n)_{n \in \mathbb{N}}$ in C in the following way.

$$\begin{aligned}
&M = 1 \rightarrow \text{ take } \mathbf{x}_1 \in C \text{ such that } |f(\mathbf{x}_1)| \geq 1, \\
&M = 2 \rightarrow \text{ take } \mathbf{x}_2 \in C \text{ such that } |f(\mathbf{x}_2)| \geq 2, \\
&\ldots, \\
&M = n \rightarrow \text{ take } \mathbf{x}_n \in C \text{ such that } f(\mathbf{x}_n) \geq n, \\
&\ldots
\end{aligned}$$

This creates a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$. The sequence takes values in the compact set C. By the Bolzano-Weierstrass theorem, it has a convergent subsequence, which we denote by $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$. Let us denote the limit of this subsequence by $\mathbf{x}$, so $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$. The domain C is closed, so we have that $\mathbf{x} \in C$.

As $\mathbf{x} \in C$, we can look at its value under the function, $f(.)$, namely $f(\mathbf{x})$. Now, given that $f(\mathbf{x}) \in \mathbb{R}$, we can find a number $M \in \mathbb{N}$ such that $|f(\mathbf{x})| < M$.[70]

[69] We use the notation $\mathbf{x}_M$ to remind that the choice of $\mathbf{x}$ may depend on the value of M.

[70] For example, M can be set as the smallest integer greater than $|f(\mathbf{x})|$.

By the construction above, we also have that for all $\varphi(n)$,

$$|f(\mathbf{x}_{\varphi(n)})| \geq \varphi(n).$$

Take the subsequence $(\mathbf{x}_{\varphi(n)})_{\varphi(n) \geq M}$.[71] This sequence is a subsequence of $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$ so it has the same limit: $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$. By continuity of the function $f(.)$, we have $f(\mathbf{x}_{\varphi(n)}) \xrightarrow{n} f(\mathbf{x})$ and consequentially, $|f(\mathbf{x}_{\varphi(n)})| \xrightarrow{n} |f(\mathbf{x})|$.[72] Also, for all $\varphi(n) \geq M$,

$$|f(\mathbf{x}_{\varphi(n)})| \geq \varphi(n) \geq M.$$

Applying Theorem 2, gives,

$$|f(x)| \geq M,$$

which gives the desired contradiction (with $f(x) < M$. This shows that $f(C)$ is bounded.[73]

Next, we need to show that $f(C)$ is closed. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $f(C)$ and assume that $y_n \xrightarrow{n} y$. We need to show that $y \in f(C)$.

By definition, $y_n \in f(C)$ means that for all $n \in \mathbb{N}$, there exists a vector $\mathbf{x}_n \in C$ with $y_n = f(\mathbf{x}_n)$. Then $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a sequence in a compact set C, so it has a convergent subsequence, say $(\mathbf{x}_{\varphi(n)})_{n \in \mathbb{N}}$. Let $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$. As C is compact, we know that $\mathbf{x} \in C$. Also $f(\mathbf{x}) \in f(C)$, by definition. Let us show that $y = f(\mathbf{x})$ thereby showing that $y \in f(C)$.

Now, $\mathbf{x}_{\varphi(n)} \to \mathbf{x}$, so by continuity of f, $f(\mathbf{x}_{\varphi(n)}) \to f(\mathbf{x})$. We also know that $f(\mathbf{x}_{\varphi(n)}) = y_{\varphi(n)}$ so

$$\lim_n y_{\varphi(n)} = \lim_n f(\mathbf{x}_{\varphi(n)}) = f(\mathbf{x}).$$

The sequence $(y_{\varphi(n)})_{n \in \mathbb{N}}$ is a subsequence of $(y_n)_{n \in \mathbb{N}}$ and $y_n \to y$. So $y_{\varphi(n)} \xrightarrow{n} y$. Given that the sequence $(y_{\varphi(n)})_{n \in \mathbb{N}}$ converges to both $f(\mathbf{x})$ and y it follows that $f(\mathbf{x}) = y$, thereby showing that $y \in f(C)$. $\square$

Let f be a continuous function with domain C and assume that C is compact. Above theorem shows that $f(C)$ is also compact. Additionally, $f(C) \subseteq \mathbb{R}$ so Theorem 12 shows that,

$$\sup f(C) \in f(C) \text{ and } \inf f(C) \in f(C).$$

In other words, there exist vectors $\mathbf{x}_0$ and $\mathbf{x}_1 \in C$ such that $f(\mathbf{x}_0) = \sup_{\mathbf{x} \in C} f(\mathbf{x})$ and $f(\mathbf{x}_1) = \inf_{x \in C} f(\mathbf{x})$. This is the main gist behind the Extreme value theorem.

Theorem 14 (Extreme value theorem)**.** *Let C be compact and f be continuous. Then f has both a maximum and minimum in C.*

[71] This is the sequence starting at $\varphi(n)$ equal to the smallest value above M.

[72] Prove this. Namely, if $(x_n)_n \in \mathbb{N}$ is a scalar sequence and $x_n \xrightarrow{n} x$ then $|x_n| \xrightarrow{n} |x|$.

[73] The short summary of the proof is the following. If $f(C)$ is unbounded, we can find a sequence in the domain whose function values are ever increasing. This sequence has a convergent subsequence, whose limit is (by continuity) larger than any possible value.

Proof. We only proof that f has a maximum. The proof of a minimum is similar.

We know that $f(C)$ is compact, so $f(C)$ has a supremum and the supremum is in $f(C)$. Let $y = \sup f(C)$. Then, by definition there is an $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = y$. Also for all $\mathbf{x} \in S$,

$$f(\mathbf{x}_0) = y \geq f(\mathbf{x}).$$

so $\mathbf{x}_0$ is a maximum. $\square$

As an example of the extreme value theorem, consider the standard consumer utility maximization model. There is a vector of prices $\mathbf{p} \in \mathbb{R}^k_{++}$ and a strict positive income level $m > 0$. The consumer chooses a bundle $\mathbf{q} \in \mathbb{R}^k_+$ to maximize a continuous utility function $u : \mathbb{R}^k_+ \to \mathbb{R}$. Here, $u(\mathbf{q})$ is the utility of consuming the bundle $\mathbf{q}$. Of course, the bundles that the consumer can choose can not be more expensive than the total income that she has. As such, the set of feasible bundles is given by,[74]

[74] This set us called the budget constraint.

$$B = \{\mathbf{q} \in \mathbb{R}^k_+ : \mathbf{p} \cdot \mathbf{q} \leq m\}.$$

The consumer then solves the following maximization problem.

Recall that $\mathbf{p} \cdot \mathbf{q} = \sum_{i=1}^k p_i q_i$ which is the total amount spend on bundle $\mathbf{q}$.

$$\max_{\mathbf{q} \in B} u(\mathbf{q}).$$

Let us show that this problem is well defined. First of all, u is continuous. So in order to apply the extreme value theorem, we only need to show that B is compact, i.e. bounded and closed. To see that B is closed, let $(\mathbf{q}_n)_{n \in \mathbb{N}}$ be a sequence of bundles in B, and assume that $\mathbf{q}_n \xrightarrow{n} \mathbf{q}$. Then for all $n \in \mathbb{N}$, $\mathbf{q}_n \in B$, so,

$$\mathbf{0} \leq \mathbf{q}_n,$$
$$\mathbf{p} \cdot \mathbf{q}_n \leq m.$$

Applying Theorem 2 component-wise to he first set of inequalities gives

$$\mathbf{0} \leq \mathbf{q}.$$

Next, Applying Theorem 2 to the sequence of scalars $(\mathbf{p} \cdot \mathbf{q}_n)_{n \in \mathbb{N}}$ gives,

$$\mathbf{p} \cdot \mathbf{q} \leq m.$$

This shows that $\mathbf{q} \in B$, so B is indeed closed. Next, we need to show that B is bounded. Let $\mathbf{q} \in B$. Let $\underline{p} = \min_k p_k > 0$ be the lowest price of all goods. Then,

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= \sum_{j=1}^k p_j q_j \leq m, \\ \to q_k &\leq \frac{m - \sum_{j \neq i} p_j q_j}{p_k} \leq \frac{m}{p_k} \leq \frac{m}{\underline{p}}. \end{aligned}$$

Then,

$$\|\mathbf{q}\| = \sqrt{\sum_{i=1}^{k} q_k^2} \leq \sqrt{\sum_{i=1}^{k} \left(\frac{m}{\underline{p}}\right)^2} = \frac{m}{\underline{p}}\sqrt{k}.$$

So we see that B is indeed bounded.[75]

This shows that the consumer optimization problem is well defined.[76] However, it does not tell you how the solution may look like nor how one might find this utility maximizing bundle.

[75] It is possible to obtain lower upper-bounds. However, in order to show that B is bounded we only need to find one.

[76] Given that the utility function is continuous on S.

Intermediate value theorem

The second result of this chapter is called the intermediate value theorem.

Theorem 15 (Intermediate value theorem)**.** *Let $a, b \in \mathbb{R}, a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for all z with,*

$$\min_{x\in[a,b]} f(x) \leq z \leq \max_{x\in[a,b]} f(x),$$

there is a $c \in [a, b]$ such that $f(c) = z$.

Proof. Let x_m minimize $f(x)$ over $[a, b]$ and let x_M maximize $f(x)$ over $[a, b]$ and let $f(x_m) \leq z \leq f(x_M)$. If $z = f(x_M)$ or $z = f(x_m)$, then we are done as $x_M, x_m \in [a, b]$. So assume that $f(x_m) < z < f(x_M)$.

Assume that $x_m < x_M$.[77] We are going to construct two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in the interval $[x_m, x_M]$ that will converge to the same point. Also, for all n, we will require that $f(a_n) \leq z$ and $f(b_n) > z$.

[77] Exercise: prove the theorem for $x_m > x_M$.

For $n = 1$, set $a_1 = x_m$ and $b_1 = x_M$. Having defined $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ take the element c to be the midpoint between a_n and b_n,

$$c = \frac{a_n + b_n}{2}.$$

There are two possibilities. If $f(c) \leq z$, we define $a_{n+1} = c$ and $b_{n+1} = b_n$. If $f(c) > z$, we define $a_{n+1} = a_n$ and $b_{n+1} = c$. Observe that

$$|a_{n+1} - b_{n+1}| = \frac{|a_n - b_n|}{2},$$

As such, after n steps we have that $|a_{n+1} - b_{n+1}| = \frac{|x_M - x_m|}{2^n} \xrightarrow{n} 0$.

In addition, the sequence $(a_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence, which is bounded from above (by x_M). Next, the sequence $(b_n)_{n\in\mathbb{N}}$ is a non-increasing sequence, which is bounded from below (by x_m). This means that both converge, say to a and b. Then also,

$$|a - b| = |a - a_n + b_n - b + a_n - b_n| \leq |a - a_n| + |b_n - b| + |a_n - b_n| \xrightarrow{n} 0.$$

The three terms on the right hand side converge all three to zero which means that $|a-b|=0$, so $a=b$.

Finally, notice that for all $n \in \mathbb{N}$, $f(a_n) \leq z$ and for all $n \in \mathbb{N}$, $f(b_n) \geq z$. As such, taking limits, we obtain $f(a) \leq z$ and $f(a) \geq z$ which shows that $f(a)=z$.

$\square$

The following is a simple result from the intermediate value theorem.

Theorem 16 (Mean value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a C^1 function.* [78] *Then there is a $c \in [a,b]$ such that,*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

[78] This means that f is differentiable and that the derivative is continuous

The derivative $f'(c)$ is defined as,

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}.$$

Proof. Given that f' is continuous and $[a,b]$ is compact, we have that the following are well defined,

$$m = \min_{x\in[a,b]} f'(x),$$
$$M = \max_{x\in[a,b]} f'(x).$$

Then, for all $x \in [a,b]$,

$$m \leq f'(x) \leq M.$$

Integrating all sides from a to b gives,

$$\int_a^b m dx \leq \int_a^b f'(x)dx \leq \int_a^b M dx,$$
$$m[b-a] \leq f(b)-f(a) \leq M[b-a],$$
$$m \leq \frac{f(b)-f(a)}{b-a} \leq M.$$

As such, by the intermediate value theorem, we see that there is a $c \in [a,b]$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

as was to be shown. $\square$

Exercises

1. Does the extreme value theorem imply that the profit maximization problem

$$\max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{w}\cdot\mathbf{x} \text{ subject to } \mathbf{x} \geq 0,$$

where $\mathbf{x} \in \mathbb{R}^k_+$, f a continuous function, $p \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^k_{++}$ has a solution. If not, give a counterexample.

2. For each of the following problems, determine whether the extreme value theorem guarantees a solution.

   - $\max_{x,y} x^2 + y^2$ subject to $x^2 + 2y^2 \leq 1$.
   - $\max f(x)$ subject to $x \geq 0$ where $f(x) = -x^2$ if $x \leq -1$ and $f(x) = -1$ if $x > -1$.
   - $\max f(x)$ subject to $x \in [0,1]$ where $f(x) = x^2$ if $0 \leq x < 1/2$ and $f(x) = 1/2$ if $1/2 \leq x \leq 1$.

3. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that for all y,

$$f(y) < ky^\alpha,$$

   for some parameter values $k > 0$ and $\alpha \in [0,1[$. Show that the problem $\max_y pf(y) - wy$ subject to $y \geq 0$ has a solution for any $p, w > 0$.

4. Let f be a continuous function on $[a,b]$. Show that there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

5. Consider the equator as a large circle and assume that temperature is a continuous function of longitude. Show that there are two points on the opposite side of the equator (and therefore the earth) that have the same temperature.

6. A polynomial of odd degree is a function $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a^k x^k$ such that k is an odd number. Show that any such polynomial has a real root, i.e. there is a number $x \in \mathbb{R}$ such that $f(x) = 0$.

Correspondences

A CORRESPONDENCE IS a generalization of a function. Remember that a function associates to each value in its domain a unique value in the range. On the other hand, a correspondence may associates to each value in the domain multiple values in the 'range'.

Definition 16 (Correspondence). *A correspondence $F : D \twoheadrightarrow R$ from a domain $D \subseteq \mathbb{R}^k$ to a set $R \subseteq \mathbb{R}^\ell$ associates to each vector $\mathbf{x} \in D$ a subset $F(\mathbf{x}) \subseteq R$.*[79]

[79] To describe a correspondence F from a domain $D \subseteq \mathbb{R}^k$ to a set $R \subseteq \mathbb{R}^\ell$, we use the notation $F : D \twoheadrightarrow R$.

AS AN EXAMPLE, let us go back to the consumer optimization model. Here, we associated to each price vector $\mathbf{p} \in \mathbb{R}_{++}$ and income level $m > 0$ a budget set,

$$B = \{\mathbf{q} \in \mathbb{R}^k_+ : \mathbf{pq} \leq m\}.$$

We can formalize this in terms of a correspondence. $B : \mathbb{R}^k_{++} \times \mathbb{R}_{++} \to \mathbb{R}^k_+$ that associates with each price vector and income level a budget set.

$$B(\mathbf{p}, m) = \{\mathbf{q} \in \mathbb{R}^k_+ : \mathbf{pq} \leq m\}.$$

Definition 17. *Let $F : D \twoheadrightarrow R$ be a correspondence. The graph of F, denoted by gr_F is given by the set,*

$$gr_F = \{(\mathbf{x}, \mathbf{y}) \in D \times R : \mathbf{y} \in F(\mathbf{x})\}.$$

If $D \subseteq \mathbb{R}^k$ and $R \subseteq \mathbb{R}^\ell$ then gr_F is a subset of $\mathbb{R}^{k+\ell}$.

PREVIOUSLY, WE DEFINED the concept of continuity for functions. When looking at correspondences, there are various way by which we can generalize this concept. We will present the two most actively used. The first is called upper hemicontinuity. The second is called lower hemicontinuity.

Definition 18. *A correspondence $F : D \twoheadrightarrow R$ is upper hemicontinuous if*

(i) for all sequences $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in D and $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in R, if

$$\mathbf{x}_n \xrightarrow{n} \mathbf{x},\ \mathbf{y}_n \xrightarrow{n} \mathbf{y},\ \mathbf{x} \in D \text{ and } \forall n \in \mathbb{N} : \mathbf{y}_n \in F(\mathbf{x}_n),\ \text{then } \mathbf{y} \in F(\mathbf{x}).$$

(ii) for all sequences $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *in* D *and* $(\mathbf{y}_n)_{n\in\mathbb{N}}$ *in* R*, if*

$$\mathbf{x}_n \xrightarrow{n} \mathbf{x},\ \mathbf{x} \in D \text{ and } \forall n \in \mathbb{N} : \mathbf{y}_n \in F(\mathbf{x}_n),\ \text{then } (\mathbf{y}_n)_{n\in\mathbb{N}} \text{ is bounded.}$$

Returning to our example, let us show that the budget set correspondence $B(\mathbf{p}, m) : \mathbb{R}^{k+1}_{++} \twoheadrightarrow \mathbb{R}^k_+$ is upper hemicontinuous. As we already showed above, $B(\mathbf{p}, m)$ is non-empty and compact for all $(\mathbf{p}, m) \in \mathbb{R}^{k+1}_{++}$.

Now, consider a sequence $((\mathbf{p}_n, m_n))_{n\in\mathbb{N}}$ with

$$(\mathbf{p}_n, m_n) \xrightarrow{n} (\mathbf{p}, m) \in \mathbb{R}^{k+1}_{++},$$

and a sequence $(\mathbf{q}_n)_{n\in\mathbb{N}} \in \mathbb{R}^k_+$ such that $\mathbf{q}_n \xrightarrow{n} \mathbf{q}$ and for all $n \in \mathbb{N}$, $\mathbf{q}_n \in B(\mathbf{p}_n, m_n)$.[80] We need to show that $\mathbf{q} \in B(\mathbf{p}, m)$.

[80] In other words, for all $n \in \mathbb{N}$, $\mathbf{q}_n \geq \mathbf{0}$ and $\mathbf{p}_n \cdot \mathbf{q}_n \leq m_n$.

First of all, for all n, we have that,

$$\mathbf{0} \leq \mathbf{q}_n.$$

Applying Theorem 2 componentwise gives $\mathbf{0} \leq \mathbf{q}$. Next, we have that for all n,

$$\mathbf{p}_n \cdot \mathbf{q}_n - m_n \leq 0.$$

It can be shown that $\mathbf{p}_n \cdot \mathbf{q}_n \xrightarrow{n} \mathbf{pq}$.[81] Together with $m_n \xrightarrow{n} m$ and applying Theorem 2, we obtain,

[81] Indeed,

$$\begin{aligned} &|\mathbf{p}_n \cdot \mathbf{q}_n - \mathbf{p} \cdot \mathbf{q}|, \\ &= |\mathbf{p}_n \cdot \mathbf{q}_n - \mathbf{p} \cdot \mathbf{q}_n + \mathbf{p} \cdot \mathbf{q}_n - \mathbf{p} \cdot \mathbf{q}|, \\ &\leq |(\mathbf{p}_n - \mathbf{p}) \cdot \mathbf{q}_n| + |\mathbf{p} \cdot (\mathbf{q}_n - \mathbf{q})|, \\ &\leq \|\mathbf{p}_n - \mathbf{p}\|\|\mathbf{q}_n\| + \|\mathbf{p}\|\|\mathbf{q}_n - \mathbf{q}\|, \\ &\to 0. \end{aligned}$$

Observe the use of the Cauchy-Schwartz inequality.

$$\mathbf{pq} - m \leq 0,$$

so $\mathbf{pq} \leq m$ which shows that $\mathbf{q} \in B(\mathbf{p}, m)$.

Next, consider a sequence $(\mathbf{p}_n, m_n) \xrightarrow{n} (\mathbf{p}, m) \in \mathbb{R}^{k+1}_{++}$ and a sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ with $\mathbf{q}_n \in B(\mathbf{p}_n, m_n)$ for all $n \in \mathbb{N}$. We need to show that the sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ is bounded. First of all observe that $(m_n)_{n\in\mathbb{N}}$ is bounded as it converges. As such, there is an M such that $m_n < M$ for all $n \in \mathbb{N}$. Also, $p^* = \inf_n(min_j p_{n,j}) > 0$ as otherwise, there would be a subsequence for which the price of one of the goods converges to 0, again a contradiction.

From this, we can conclude that $q_{n,j} \leq M/p^*$ for all $n \in \mathbb{N}$ and goods j, so the sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ is bounded.

Next, we have the definition of lower-hemi-continuity,

Definition 19. *A correspondence* $F : D \twoheadrightarrow R$ *is lower hemicontinuous if for any sequence* $(\mathbf{x}_n)_{n\in\mathbb{N}}$ *in* D *with* $\mathbf{x}_n \xrightarrow{n} \mathbf{x} \in D$ *and every* $\mathbf{y} \in F(\mathbf{x})$*, there is a sequence* $(\mathbf{y}_n)_{n\in\mathbb{N}}$ *such that* $\mathbf{y}_n \xrightarrow{n} \mathbf{y} \in F(\mathbf{x})$ *and a number* $M \in \mathbb{N}$ *such that for all* $n \geq M$*,* $\mathbf{y}_n \in F(\mathbf{x}_n)$*.*

Returning to our example, let us show that the budget set correspondence $B(\mathbf{p}, m)$ is also lower-hemi-continuous. Let $\mathbf{q} \in B(\mathbf{p}, m)$

and $(\mathbf{p}_n, m_n) \xrightarrow{n} (\mathbf{p}, m)$. We need to show that there is an $M \in \mathbb{N}$ and sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ such that $\mathbf{q}_n \xrightarrow{n} \mathbf{q}$ and for all $n \geq M$ $\mathbf{q}_n \in B(\mathbf{p}_n, m_n)$.

If $\mathbf{q} = \mathbf{0}$ then we can take $\mathbf{q}_n = \mathbf{0}$ for all $n \in \mathbb{N}$ as $\mathbf{0} \in B(\mathbf{p}_n, m_n)$ for all $n \in \mathbb{N}$. As such, assume that $\mathbf{q} \neq \mathbf{0}$.

The problem is to look for a sequence $\mathbf{q}_n \in B(\mathbf{p}_n, m_n)$ that converges to $\mathbf{q}$. We do this by setting $\mathbf{q}_n = \alpha_n \mathbf{q}$ and try to pick α_n as close to 1 as possible under the constraint that $\alpha_n \mathbf{q} \in B(\mathbf{p}_n, m_n)$.[82] In other words, we require,

[82] In other words, we rescale $\mathbf{q}$ as little as possible such that $\alpha_n \mathbf{q}$ is in the budget set $F(\mathbf{p}_n, m_n)$.

$$\mathbf{p}_n \cdot (\alpha_n \mathbf{q}) \leq m_n.$$

If $\mathbf{p}_n \cdot \mathbf{q} \leq m_n$ we can set $\alpha_n = 1$. Otherwise, if $\mathbf{p}_n \cdot \mathbf{q} > m_n$, we can set,

$$\alpha_n = \frac{m_n}{\mathbf{p}_n \cdot \mathbf{q}} < 1.$$

Summarizing, we define:

$$\mathbf{q}_n = \alpha_n \mathbf{q} \text{ where } \alpha_n = \min\left\{1, \frac{m_n}{\mathbf{p}_n \cdot \mathbf{q}}\right\}.$$

We would like to show that for this choice $(\alpha_n \mathbf{q}) \equiv \mathbf{q}_n \xrightarrow{n} \mathbf{q}$. In other words, $\alpha_n \xrightarrow{n} 1$. Notice that the function

$$\min\left\{1, \frac{m_n}{\mathbf{p}_n \cdot \mathbf{q}}\right\}$$

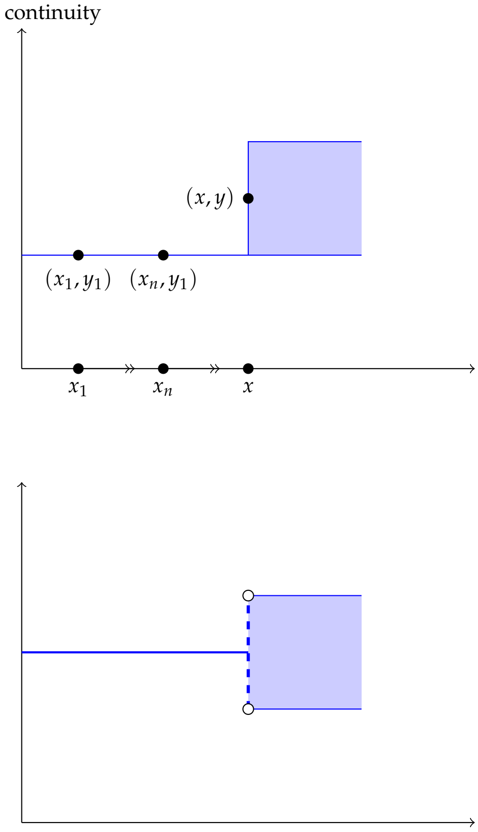
is a continuous function of m_n and $\mathbf{p}_n$.[83] As such, by continuity:

[83] Notice that we can write $\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$.

$$\alpha_n \xrightarrow{n} \min\left\{1, \frac{m}{\mathbf{p} \cdot \mathbf{q}}\right\} = 1.$$

The top part of Figure 4 shows a picture of a correspondence that is not lower-hemi-continuous. There is a sequence $x_n \xrightarrow{n} x$ and we have that $y \in F(x)$. however there is no sequence $y_n \to y$ such that $y_n \in F(x_n)$ for all n above some threshold $N \in \mathbb{N}$. The bottom part shows an example of a correspondence which is lower hemi-continuous but not upper-hemi-continuous. It is clear to see that the graph is not closed.

Figure 4: Upper and Lower-hemi-continuity



Intuitively, upper-hemi-continuity is compatible only with discontinuities that appear as explosions of sets. Lower hemi-continuity is compatible only with implosions of sets.

Definition 20. *A correspondence $F : D \twoheadrightarrow R$ is continuous if it is both upper and lower hemi-continuous.*

Exercises

1. Let $f : S \to \mathbb{R}$ be a function and consider the correspondence, $F(\mathbf{x}) = \{y : y = f(\mathbf{x})\}$. Show that if F is lower hemicontinuous at $\mathbf{x}$ or upper hemicontinuous at $\mathbf{x}$ then f is continuous at $\mathbf{x}$.

2. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be two bounded continuous functions such that for all $\mathbf{x} \in S$, $f(\mathbf{x}) \le g(\mathbf{x})$. Assume that S is compact. Show that the correspondence $G(\mathbf{x}) = \{y \in \mathbb{R} : f(\mathbf{x}) \le y \le g(\mathbf{x})\}$ is both upper and lower hemicontinuous.

3. Let $G : S \twoheadrightarrow T$ and $F : S \twoheadrightarrow T$ be two correspondences and define $K(\mathbf{x}) = \{\mathbf{y} \in G(\mathbf{x}) \cap F(\mathbf{x})\}$. Assume that for all $\mathbf{x} \in S$, $K(\mathbf{x}) \neq \emptyset$. Show that if G and F are upper hemicontinuous then K is also upper hemi-continuous.

4. Show that if the graph of a correspondence G is open and non-empty for all $\mathbf{x}$, then G is lower hemicontinuous.

Berge's maximum theorem

In 1994 Claude Berge wrote a 'mathematical' murder mystery for Oulipo.[84] In this short story Who killed the Duke of Densmore (1995),[85] the Duke of Densmore has been murdered by one of his six mistresses, and Holmes and Watson are summoned to solve the case. Watson is sent by Holmes to the Duke's castle but, on his return, the information he conveys to Holmes is very muddled. Holmes uses the information that Watson gives him to construct a graph. He then applies a theorem of György Hajós to the graph which produces the name of the murderer.

J. J. O'Connor and E. F. Robertson (cfr:http://www-history.mcs.st-andrews.ac.uk/Biographies/Berge.html)

[84] Oulipo was a group of writers and mathematicians aiming at exploring in a systematic way formal constraints on the production of literary texts.

[85] See https://jacquerie.github.io/duke/ for more for a statement of the problem. Can you solve it?

In the previous chapter we saw how a correspondence can be used to model the budget set of the consumer utility maximization problem. This approach is much more general. Berge's theorem deals with optimization problems of the following form,

$$\max_{\mathbf{x}\in G(\theta)} f(\mathbf{x},\theta).$$

where $f(\mathbf{x},\theta)$ is a function and $G(\theta)$ is a correspondence. This problem tries to find the 'optimal' vector $\mathbf{x}$,[86] given the feasibility constraint, $\mathbf{x} \in G(\theta)$. The vector $\mathbf{x}$ is called the choice variable while the vector θ is called the parameter of the choice problem. We know that when $G(\theta)$ is compact and if $f(\mathbf{x},\theta)$ is continuous in $\mathbf{x}$, then this problem has a solution. Berge's theorem further investigates the properties of this solution.

[86] In the sense of optimizing the function $f(\mathbf{x},\theta)$.

Example:

> Let $B(\mathbf{p},m) = \{\mathbf{q} \in \mathbb{R}_+ : \mathbf{p}\cdot\mathbf{q} \leq m\}$ be the budget correspondence and let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function. The consumer optimization problem takes the form,
>
> $$\max_{\mathbf{q}\in B(\mathbf{p},m)} u(\mathbf{q}).$$

Theorem 17 (Berge's maximum theorem)**.** *Let $X \subseteq \mathbb{R}^k$ and $T \subseteq \mathbb{R}^\ell$ and consider a continuous function $f : X \times T \to \mathbb{R}$. Let $G : T \twoheadrightarrow X$ be a compact* *valued correspondence. For $\theta \in T$, define*

$$v(\theta) = \max_{\mathbf{x} \in G(\theta)} f(\mathbf{x}, \theta),$$
$$\Gamma(\theta) = \{\mathbf{x} \in G(\theta) : f(\mathbf{x}, \theta) = v(\theta)\} = \arg\max_{\mathbf{x} \in G(\theta)} f(\mathbf{x}, \theta).$$

If $G : T \twoheadrightarrow X$ is continuous, then v is also continuous and $\Gamma : T \twoheadrightarrow X$ is non-empty and upper hemicontinuous.[87]

[87] Notice, G is a correspondence, v is a function and Γ is also a correspondence.

Proof. First of all, given that f is continuous and $G(\theta)$ is compact valued, we obtain by the extreme value theorem, that the optimization problem is well defined. This also implies that $\Gamma(\theta)$ is non-empty for all $\theta \in T$.

Let us first show that $v : T \to \mathbb{R}$ is a continuous function. By Lemma 3 we only need to show that any sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n \xrightarrow{n} \theta$ has a subsequence $(\theta_{\varphi(n)})_{n \in \mathbb{N}}$ such that $v(\theta_{\varphi(n)}) \xrightarrow{n} v(\theta)$.

As such, let $\theta_n \xrightarrow{n} \theta$. For every $n \in \mathbb{N}$, we can find a vector $\mathbf{x}_n \in \Gamma(\theta_n)$ that solves the maximization problem. This produces a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in X with the property that $\mathbf{x}_n \in \Gamma(\theta_n)$ for all $n \in N$.

This means that $\mathbf{x}_n \in G(\theta_n)$ for all $n \in \mathbb{N}$. Then, by upper hemicontinuity of G, the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is bounded. This means that there is a convergent subsequence $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$ of $(\mathbf{x}_n)_{n \in \mathbb{N}}$. As $\theta_{\varphi(n)} \xrightarrow{n} \theta$, $\mathbf{x}_{\varphi(n)} \xrightarrow{n} \mathbf{x}$ and for all n, $\mathbf{x}_{\varphi(n)} \in G(\theta_{\varphi(n)})$ it follows again by upper-hemicontinuity of G that $\mathbf{x} \in G(\theta)$.

Next, by continuity of f,

$$v(\theta_{\varphi(n)}) = f(\mathbf{x}_{\varphi(n)}, \theta_{\varphi(n)}) \xrightarrow{n} f(\mathbf{x}, \theta).$$

Recall that we need to show that $v(\theta_{\varphi(n)}) \xrightarrow{n} v(\theta)$, which is, given above, equivalent to showing that $f(\mathbf{x}, \theta) = v(\theta)$. The proof is by contradiction.

Assume $\mathbf{x} \notin \Gamma(\theta)$. Then, as $\Gamma(\theta)$ is non-empty, there is a vector $\mathbf{y} \in \Gamma(\theta)$ which has the property that $f(\mathbf{y}, \theta) > f(\mathbf{x}, \theta)$. Now, we can use lower hemicontinuity of G together with $\mathbf{y} \in G(\theta)$ and $\theta_{\varphi(n)} \xrightarrow{n} \theta$ to find a number M and a sequence $\mathbf{y}_{\varphi(n)} \to \mathbf{y}$ such that $\mathbf{y}_{\varphi(n)} \in G(\theta_{\varphi(n)})$ for all $\varphi(n) \geq M$.

Also, given that $\mathbf{x}_{\varphi(n)} \in \Gamma(\theta_{\varphi(n)})$, we have that for all $n \in \mathbb{N}$:

$$f(\mathbf{x}_{\varphi(n)}, \theta_{\varphi(n)}) \geq f(\mathbf{y}_{\varphi(n)}, \theta_{\varphi(n)}).$$

taking limits, we have that, by continuity of f:

$$f(\mathbf{x}, \theta) \geq f(\mathbf{y}, \theta),$$

a contradiction. From this, we conclude that v is continuous.

Let us now show upper hemicontinuity of Γ. Let $(\theta_n)_{n\in\mathbb{N}}$ in T and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X be such that $\theta_n \xrightarrow{n} \theta \in T$, $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$ and for all $n \in \mathbb{N}$, $\mathbf{x}_n \in \Gamma(\theta_n)$. Then,

$$f(\mathbf{x},\theta) = \lim_n f(\mathbf{x}_n,\theta_n) = \lim_n v(\theta_n) = v(\theta).$$

The first equality follows from continuity of f. The last equality follows from continuity of v. This shows that $x \in \Gamma(\theta)$.

Next, let $(\theta_n)_{n\in N}$ in T and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X such that $\theta_n \xrightarrow{n} \theta \in T$ and for all $n \in \mathbb{N}$, $\mathbf{x}_n \in \Gamma(\theta_n)$. Then for all $n \in \mathbb{N}$, $\mathbf{x}_n \in G(\theta_n)$. As G is upper hemi-continuous, it follows that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is bounded. $\square$

Above theorem shows that under the stated conditions, the optimal value correspondence $\Gamma(\theta)$ is upper hemicontinuous. It is not necessarily the case that the correspondence $\Gamma(\theta)$ is also lower-hemi-continuous. Consider, for example, the following problem,

$$v(\theta) = \max_{x\in[0,1]} \theta x + (1-\theta)(1-x).$$

The solution correspondence is given by,

$$\Gamma(\theta) = \{x \in [0,1] : \theta x + (1-\theta)(1-x) = v(\theta)\}.$$

Here $G(\theta) = [0,1]$ for all θ and $f(x,\theta) = \theta x + (1-\theta)(1-x)$. It is easily established that all conditions of the theorem are valid. The optimal solution $\Gamma(\theta)$ is given by,

$$\Gamma(\theta) = \begin{cases} 0 & \text{if } \theta < 1/2, \\ [0,1] & \text{if } \theta = 1/2, \\ 1 & \text{if } \theta > 1/2 \end{cases}$$

One easily confirms that this correspondence is upper hemi-continuous but not lower hemicontinuous. Indeed, we have that $1/2 - 1/t \to 1/2$ and that $1/2 \in \Gamma(1/2)$ but for all $t > 0$, $\Gamma(1/2 - 1/t) = \{0\}$ and $0 \not\to 1/2$.

The continuity of the function $v(\theta)$ implies that if the parameters of the optimization problem, i.e. θ changes a little bit, then the optimal value function only changes slightly. On the other hand, $\Gamma(.)$ is only upper hemicontinuous, so it can be that this small change in θ produces a large change in the optimal solution for the problem. However, the various solutions (for varying θ) will only have a small effect on the value function $f(.,\theta)$ solution of the problem. In order to have that a small variation in θ leads to a small variation in the optimal solution, we need more conditions. This is the subject of the next section.

Exercises

1. Let $T = \mathbb{R}$ and let $G(\theta) = Y = [-1, 1]$ for all $\theta \in T$. Define $f : T \times X \to \mathbb{R}$ by
$$f(x, \theta) = \theta x^2.$$
Graph $\Gamma(\theta)$ and show that $\Gamma(\theta)$ is upper hemicontinuous but not lower hemicontinuous at $\theta = 0$.

2. Let $T = \mathbb{R}$ and let $G(\theta) = [0, 4]$ for all $\theta \in T$. Define,
$$f(x, \theta) = \max[2 - (x-1)^2, \theta + 1 - (x-2)^2],$$
Graph $\Gamma(\theta)$ and show that it is upper hemicontinuous. Where does it fail to be lower hemicontinuous?

3. let $T = [0, 2\pi]$ and $G(\theta) = \{x \in \mathbb{R} : -\theta \leq x \leq \theta\}$, and $f(x, \theta) = \cos(x)$. Graph $\Gamma(\theta)$ and show that it is upper hemicontinuous. Where does it fail to be lower hemicontinuous?

Convexity

Figure 5: Convex sets

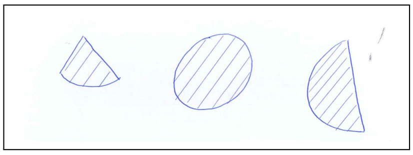
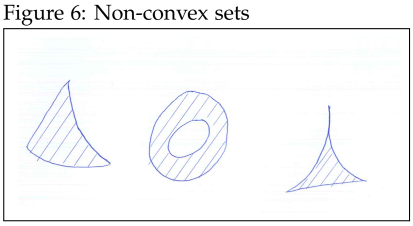


Figure 6: Non-convex sets



Geometrically speaking, a set S is convex if for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in S, every point on the line segment between $\mathbf{x}$ and $\mathbf{y}$ is also in S. Formally, it is defined in the following way.

Definition 21 (Convexity)**.** *A set $S \subseteq \mathbb{R}^n$ is* ***convex*** *if for all vectors $\mathbf{x}, \mathbf{y} \in S$ and all $\alpha \in [0,1]$, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S$.*

Let D be a convex set. Given a real valued function $f : D \to \mathbb{R}$, we can define it's epigraph,

$$\text{epi} f = \{(\mathbf{x}, y) : y \geq f(\mathbf{x})\}.$$

A function is said to be convex, if the epigraph of f is a convex set. The hypograph of the function $f : D \to \mathbb{R}$ is given by

$$\text{hypo} f = \{(\mathbf{x}, y) : y \leq f(\mathbf{x})\}.$$

A function is concave if the hypograph of f is a convex set. Rephrasing these conditions in terms of inequalities gives the following,

Definition 22 (convexity, concavity)**.** *Let D be a convex set and let $f : D \to \mathbb{R}$ be a real valued function. Then,*

- *f is* ***convex*** *if for all, $\mathbf{x}_1, \mathbf{x}_2 \in D$ and $y_1 \geq f(\mathbf{x}_1), y_2 \geq f(\mathbf{x}_2)$ and $\alpha \in [0,1]$,*
$$\alpha y_1 + (1-\alpha) y_2 \geq f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2).$$
- *f is* ***strictly convex*** *if f is convex and for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ and for all $y_1 \geq f(\mathbf{x}_1)$, $y_2 \geq f(\mathbf{x}_2)$ and all $\alpha \in (0,1)$,*
$$\alpha y_1 + (1-\alpha) y_2 > f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2).$$
- *f is* ***concave*** *if for all $\mathbf{x}_1, \mathbf{x}_2 \in D$, and $y_1 \leq f(\mathbf{x}_1)$, $y_2 \leq f(\mathbf{x}_2)$ and $\alpha \in [0,1]$,*
$$\alpha y_1 + (1-\alpha) y_2 \leq f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2).$$

- f *is* ***strictly concave*** *if* f *is concave and in addition for all* $\mathbf{x}_1, \mathbf{x}_2 \in D$ *with* $\mathbf{x}_1 \neq \mathbf{x}_2$ *and* $y_1 \leq f(\mathbf{x}_1), y_2 \leq f(\mathbf{x}_2)$ *and all* $\alpha \in (0,1)$*,*
$$\alpha y_1 + (1-\alpha) y_2 < f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2).$$

For D a convex set, let $f : D \to \mathbb{R}$ be a real valued function. Define it's lower level set,
$$Lf_\alpha = \{\mathbf{x} \in D : f(\mathbf{x}) \leq \alpha\}.$$
A function is quasi-convex if for all α, the sets Lf_α are convex. The upper level set of a function f is given by,
$$Uf_\alpha = \{x \in D : f(\mathbf{x}) \geq \alpha\}.$$
A function is quasi-concave if for all α, the upper level sets Uf_α are convex.

Definition 23 (quasi-convexity, quasi-concavity)**.** *Let* D *be a convex set and let* $f : D \to \mathbb{R}$ *be a real valued function. Then,*

- f *is* ***quasi-convex*** *if for all* $\mathbf{x}_1, \mathbf{x}_2 \in D$*, and* $y \geq f(\mathbf{x}_1), f(\mathbf{x}_2)$ *and* $\alpha \in [0,1]$*,*
$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq y.$$
- f *is* ***strictly quasi-convex*** *if* f *is quasi-convex and in addition for all* $\mathbf{x}_1, \mathbf{x}_2 \in D$ *with* $\mathbf{x}_1 \neq \mathbf{x}_2$ *and all* $y \geq f(\mathbf{x}_1), f(\mathbf{x}_2)$ *and all* $\alpha \in (0,1)$*,*
$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < y.$$
- $f : S \to \mathbb{R}$ *is* ***quasi-concave*** *if for all* $\mathbf{x}_1, \mathbf{x}_2 \in D$ *and* $y \leq f(\mathbf{x}_1), f(\mathbf{x}_2)$ *and all* $\alpha \in [0,1]$*,*
$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq y.$$
- f *is* ***strictly quasi-concave*** *if* f *is quasi-concave and in addition for all* $\mathbf{x}_1, \mathbf{x}_2 \in D$ *with* $\mathbf{x}_1 \neq \mathbf{x}_2$ *and all* $y \geq f(\mathbf{x}_1), f(\mathbf{x}_2)$ *and all* $\alpha \in (0,1)$*,*
$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) > y.$$

Quasi-concavity or quasi-convexity is a weaker notion than convexity or concavity.

Lemma 4. *If* f *is convex, then* f *is quasi-convex. If* f *is strictly convex, then* f *is strictly quasi-convex. If* f *is concave, then* f *is quasi-concave and if* f *is strictly concave, then* f *is also strictly quasi-concave.*

Proof. Here we only show that convexity implies quasi-convexity.[88] Let f be convex and let $f(\mathbf{x}_1), f(\mathbf{x}_2) \leq y$. Then $y \geq f(\mathbf{x}_1)$ and $y \geq f(\mathbf{x}_2)$ so by convexity,
$$\alpha y + (1-\alpha) y = y \geq f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2),$$
as was to be shown. The strict and concave versions follows immediately. □

[88] Show the other implications yourself.

Definition 24 (Convex valued correspondence)**.** *A correspondence $G : S \twoheadrightarrow T$ is convex valued if for all $\mathbf{x} \in S$, $G(\mathbf{x})$ is a convex set.*

The following theorem imposed additional assumptions on Berge's theorem in order to guarantee that the correspondence $\Gamma(\theta)$ is a function. First, we need that the correspondence G is convex valued. Second, it assumes that the objective function f is strictly quasi-concave.

Theorem 18. *Let, X and T be two convex sets, let $f : X \times T \to \mathbb{R}$ be a continuous function and let $G : T \twoheadrightarrow X$ be a continuous, correspondence. Let,*

$$v(\theta) = \max_{\mathbf{x} \in G(\theta)} f(\mathbf{x}, \theta),$$
$$\Gamma(\theta) = \{\mathbf{x} \in S : f(\mathbf{x}, \theta) = v(\theta)\}.$$

If $f(\mathbf{x}, \theta)$ is quasi-concave in $\mathbf{x}$ and G is convex valued, then Γ is also convex valued. If in addition f is strictly quasi-concave, then Γ is single valued.[89].

[89] In particular, a continuous function

Proof. Let f be concave in $\mathbf{x}$ and let G be convex valued. We need to show that $\Gamma(\theta)$ is a convex set for all θ. If $\mathbf{x}_1, \mathbf{x}_2 \in \Gamma(\theta)$ then we have that,

$$f(\mathbf{x}_1, \theta) = v(\theta) = f(\mathbf{x}_2, \theta).$$

As such,

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \theta) \geq v(\theta).$$

Also, as $G(\theta)$ is convex, $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in G(\theta)$. So $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_1 \in \Gamma(\theta)$ as was to be shown.

For the second part, if $\mathbf{x}_1 \neq \mathbf{x}_2$ then,

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \theta) > v(\theta).$$

However, this is impossible as $v(\theta)$ is the optimal value function. As such, it must be that $\mathbf{x}_1 = \mathbf{x}_2$ and therefore $\Gamma(\theta)$ must be unique.

The correspondence $\Gamma(\theta)$ is upper hemi-continuous, so if it is single valued, we have by Exercise 1 of the chapter on Corresondences that it coincides with the graph of a continuous function. ☐

Exercises

1. Show that a function f is (strictly) convex if and only if $-f$ is (strictly) concave.

2. Show that a function f is (strictly) quasi-convex if and only if $-f$ is (strictly) quasi-concave.

3. Show that above theorem also holds if max is replaced by min and (strict) quasi-concave by (strict) quasi-convex.

Sperner's lemma

The major fixed point theorem is due to Brouwer. There are several proofs of Brouwer's fixed point with various degrees of difficulty. The easiest[90] relies on an intermediate result that is called Sperner's lemma. In order to introduce this lemma, it is necessary to provide some additional notation and definitions.

[90] Probably.

A **polyhedral** is the convex combination of a finite number of vectors. An n-dimensional simplex, denoted by S_n is a polyhedral which is the convex combination of $n+1$ affinely independent vectors.[91] An n dimensional simplex lies in an n dimensional plane of $\mathbb{R}^{n+1}$.

[91] The vectors $\mathbf{x}_0, \ldots, \mathbf{x}_n$ are affinely independent if $\{\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0\}$ are linearly independent. In other words, if $\sum_{i=1}^n \alpha_i(\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}$ only if all $\alpha_i = 0$. As an example, two vectors are affinely independent if they are distinct. In two dimensional space, 3 vectors are affinely independent if they form the vertices of a triangle.

Example:

- S_1 is the one dimensional simplex. It is formed by the convex combination of 2 linearly independent vectors. This means that S_1 is a line segment.
- S_2 is the 2 dimensional simplex. It is formed by the convex combination of 3 linearly independent vectors, which means that S_2 is a triangle.
- S_3 is the 3-dimensional simplex. it is formed by the convex combination of 4 linearly independent vectors, so it is a tetahedron.

For these lecture notes, we will look at S_1 and S_2, i.e. line segments and triangles. Sperner's lemma, however, also extends to higher dimensions.

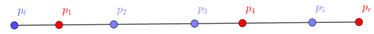
Consider a one dimensional simples S_1, better known as a line segment. Let us denote the left endpoint of the segment by p_ℓ and the right endpoint by p_r. It is fairly easy to subdivide this line segment into smaller segments. This can be done by picking points $p_1, p_2, \ldots, p_n$ on the line segment (p_ℓ, p_r). Giving the smaller segments,

$$(p_\ell, p_1), (p_1, p_2), (p_3, p_4), \ldots, (p_n, p_r).$$

Dividing the segment into smaller segments is called a simplical subdivision. The points $p_\ell, p_1, \ldots, p_n, p_r$ are called the vertices of the subdivision.

Let us now colour these vertices. For a one dimensional simplex, we pick two colours, say red and blue and we may colour the vertices anyway we want except for the fact that p_ℓ and p_r have different colours. Such colouring is called a Sperner colourling.

Figure 7: A Sperner colouring of S_1



Theorem 19 (Sperner)**.** *Let S_1 be a one-dimensional simplex which is subdivided. Assume that all points are coloured in one of two colours and that p_ℓ and p_r have different colours (i.e. a Sperner colouring). Then there are an odd number of sub-segments for which both vertices have different colours.*

Proof. Use the following analogy. Let the line segment be a building and let the points $p_\ell, p_1, \ldots, p_n, p_r$ be the walls of the building. A subsegment (p_i, p_{i+1}) is a room in the building.[92] Let's put a door in every wall that has the color red. And before every door we put a doormat, but only when we are inside the room.[93]

[92] This is a one-dimensional building, so a room has only two walls.

[93] In other words, we don't put doormats outside the building.

How many doormats are there? Well we can count the number of doormats in two different ways. First of all, according to the doors in our building. We have one door that goes to the ouside,[94] Next, for every door on the inside, we have to put two doormats as each such door connects two rooms. As such,

[94] There is the door in wall $\mathbf{p}_\ell$ or p_r according to which one of the two has the colour red.

$$\# \text{ mats} = 2 \ (\# \text{ inside doors }) + 1.$$

Next, let us count the number of mats according to the rooms. There are three kind of rooms. Rooms where both walls are red. Rooms where both walls are blue and rooms where one wall is red and one is blue. The first type of room has 2 doormats (as there are two doors). The second type of door has no doormats. The third type has one doormat. As such,

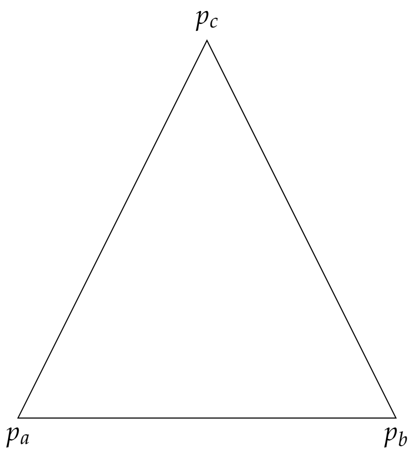
$$\# \text{ mats} = 2 \ (\# \text{ red-red room}) + (\# \text{ red-blue room}).$$

Equating the two ways of counting the doormats gives,

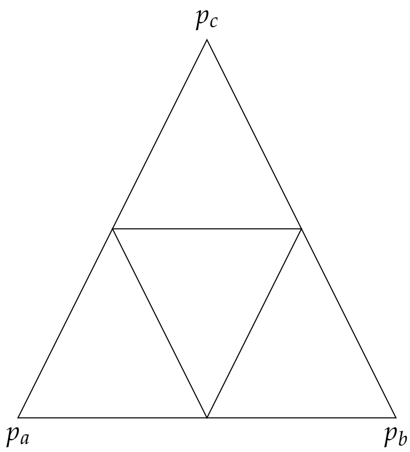
$$\begin{aligned} 2 \ (\# \text{ inside doors }) + 1 &= 2 \ (\# \text{ red-red room}) + (\# \text{ red-blue room}), \\ \leftrightarrow (\# \text{ red-blue room}) &= 2 \left((\# \text{ inside doors }) - (\# \text{ red-red room}) \right) + 1. \end{aligned}$$

The right hand side is an even number plus an odd number, so it is odd. This means that the number of red-blue rooms should also be odd, in particular, there should be at least one. □

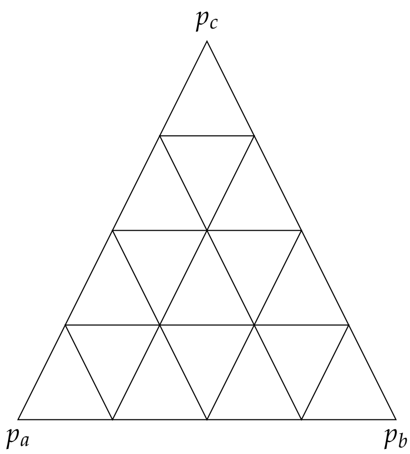
Let us now have a look at the two dimensional simplex S_2, which is a triangle. First, we need to divide the triangle into smaller triangles. There are many ways to do this but the following will work. Take a triangle,



divide each side of the triangle in half and connect these points. This gives four smaller triangles.



Each side of this smaller triangle can again be divided in four smaller ones.

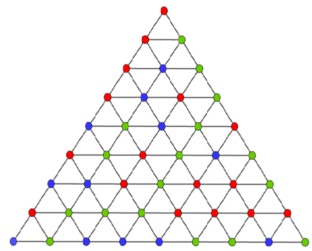


and so on ....

Each time we get smaller and smaller triangles. Let p_a, p_b and p_c be the three vertices of the big triangle and let $p_1, p_2, \ldots p_n$ be all other vertices in the triangulation. We will denote the big triangle by T and the small triangles by $T_1, T_2, \ldots, T_m$. We will colour the vertices in 3 colours. The rules are the following

- The three vertices of the big triangle p_a, p_b, p_c have different colours.
- If a vertex p_i is on one of the sides of the big triangle, it needs to have the colour of one of the vertices of the big triangle that makes this side. For example if p_i is on the line segment made up by p_a and p_b then p_i must have the colour of p_a or the colour of p_b.

Figure 8: Sperner colouring of S_2



Such colouring is called a Sperner colouring.

Theorem 20. *Consider a triangulation of which the vertices are coloured according to the rules above, i.e. a Sperner colouring. Then there are an odd number of small triangles T_i of which the three vertices that make up these triangles all have distinct colours.*

Proof. Let the colors be red, blue and green. Assume, wlog that p_a is red, p_b blue and p_c green.

We will, once more, use the building analogy. Let the big triangle be our building. A room is now a small triangle and (obviously) every room now has three walls, where each wall is defined by two vertices of the small triangle. These vertices which may or may not have different colours, so every wall is defined by two (possibly identical) colours.

Pick two of the three colours, say red and blue and let us put a door into every wall with these colours. Finally, we put a doormat in front of all doors, but only on the inside of the building.

As in the previous Theorem, we will count the number of doormats in two different ways, according to the number of doors and once according to the number of rooms.

Concerning the second option, there are three types of rooms. Rooms with the colours red-blue-green have one single door (in the red-blue wall). Rooms with colours red-red-blue and red-blue-blue have two doors as there are two red-blue walls. Therefore, they also have two doormats. Finally, rooms with colours red-red-green; red-green-green; blue-blue-green or blue-green-green have no doors, so no doormats. Given this, the total number of doormats is given by

$$
\begin{aligned}
\#\text{ mats} = &\#\text{ red-blue-geen rooms,} \\
&+ 2\ \#\text{ red-blue-blue rooms,} \\
&+ 2\ \#\text{ red-red-blue rooms.}
\end{aligned}
$$

Next, let us count the doormats according to the number of doors. First, let us look at the outside of the building. There are no doors on the (p_a, p_c) nor (p_b, p_c) segments as these vertices only have the colours red-green or green-blue.

As such, all doors are on the line segment going from p_a to p_b. All walls on this segment have vertices that are coloured with red or blue. As such, by the previous Theorem, there are an odd number of (walls) line segments with distinct colours which are exactly the walls with the doors. As such, there are an odd number of doors on the outside of the big triangle. Each of these doors have exactly one doormat.

Next, for every door on the inside of the triangle, these doors connect two rooms. So there are two doormats for each inside door. As such,

$$\begin{aligned}\text{\# mats} =& \text{\# doors on boundary of big triangle,} \\ &+ 2 \text{ \# doors in interior.}\end{aligned}$$

Equating the two equations gives,

$$\begin{aligned}\text{\# red-blue-geen rooms} =& 2 \text{ \# doors in interior,} \\ &- 2 \text{ \# red-blue-blue rooms,} \\ &- 2 \text{ \# red-red-blue rooms,} \\ &+ \text{\# doors on boundary of big triangle.}\end{aligned}$$

The right hand side of this equation is odd, so the left hand side should also be odd. In particular, there is at least one room with three distinct colours. □

Exercises

1. Prove the following grid version of Sperner's lemma: Consider a 3-coloring over $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$. The coloring is proper if

   - The color of $(0, i)$ is 'red' for all $i \in \{0, 1, \ldots, n\}$.
   - The color of $(i, 0)$ is 'green' for all $i \in \{1, \ldots, n\}$. and
   - All other points on the boundary of the grid are blue.

   Show that any proper 3-coloring has a unit-size square whose vertices have all three colors.

Brouwer's fixed point theorem

The Brouwer fixed point theorem is one of the most fascinating and important theorems of 20th century mathematics. Proving this theorem established Brouwer as one of the preeminent topologists of his day. But he refused to lecture on the subject, and in fact he ultimately rejected this (his own!) work. The reason for this strange behavior is that Brouwer had become a convert to constructivism or intuitionism. He rejected the Aristotelian dialectic that a statement is either true or false and there is no alternative, so he rejected the concept of "proof by contradiction".

(Stefen G. Krantz in the history of Mathematics, edited by Lundsgaard and Gray)

The major fixed point theorem is due to Brouwer.

Theorem 21 (Brouwer)**.** *If S is compact and convex and $f : S \to S$ is continuous then f has a fixed point in S.*

We will first proof the theorem for a particular kind of convex compact set, namely the $n-1$-dimnesional unit simplex. It is given by,

$$\Delta^{n-1} = \left\{ \mathbf{x} \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j = 1 \right\}.$$

The set Δ^{n-1} consists of all non-negative n-dimensional vectors whose components sum to one. For Δ^1 these are the vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1, x_2 \geq 0$ and $x_1 + x_2 = 1$. In other words, the vectors on the line segment connecting $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Notice that this is simply a one-dimensional object, namely a line segment. As such, we can see this as being a one-dimensional simplex.

The two dimensional unit simplex Δ^2 consists of the three dimensional vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 1$. These are the vectors that lie in the triangle bounded by the corners $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

It is simply a triangle, so a 2 dimensional simplex.

Figure 9: The one dimensional unit simplex

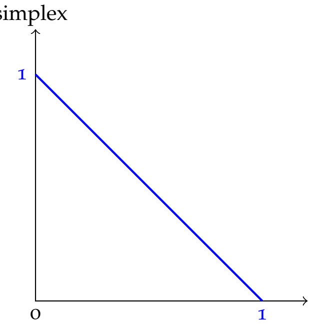
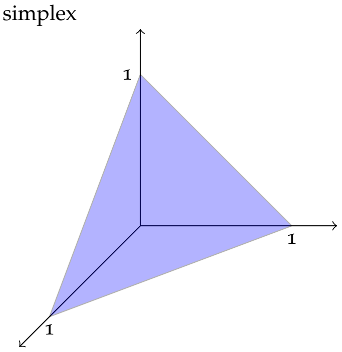


Figure 10: The two dimensional unit simplex



The following theorem shows that any continuous function from Δ^n to itself has a fixed point.

Theorem 22. *Let $f : \Delta^n \to \Delta^n$ be a continuous function. Then there is an $\mathbf{x} \in \Delta^n$ such that $f(\mathbf{x}) = \mathbf{x}$.*

Proof. We will prove it for the two dimensional simplex Δ^2. Remember, Δ^2 consists of 3-dimensional non-negative vectors whose components add up to one. The proof uses Sperner's lemma from the previous chapter.

Consider Δ^2, the two dimensional unit simplex. Let $f : \Delta^2 \to \Delta^2$ be a continuous function. We will show that f has a fixed point. The proof is by contradiction, so assume that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in \Delta^2$.

Consider a triangulation Δ^2 into small triangles. We will colour the vertices of this triangulation in the following way. Consider three colours, red, blue and green. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be a vertex of the triangulation.[95]

Then we can put,

$$f(\mathbf{v}) = \begin{bmatrix} f_1(\mathbf{v}) \\ f_2(\mathbf{v}) \\ f_3(\mathbf{v}) \end{bmatrix}$$

we know that $f(\mathbf{v}) \in \Delta^2$ so, $f_1(\mathbf{v}), f_2(\mathbf{v}), f_3(\mathbf{v}) \geq 0$ and $f_1(\mathbf{v}) + f_2(\mathbf{v}) + f_3(\mathbf{v}) = 1$. We give $\mathbf{v}$ the following colour,

$$\begin{aligned} &f_1(\mathbf{v})) < v_1 \to \text{ red,} \\ &f_1(\mathbf{v}) \geq v_1 \text{ and } f_2(\mathbf{v}) < v_2 \to \text{ blue,} \\ &f_1(\mathbf{v}) \geq v_1 \text{ and } f_2(\mathbf{v}) \geq v_2 \text{ and } f_3(\mathbf{v}) < v_3 \to \text{ green.} \end{aligned}$$

First of all, let us show that we can actually colour each vertex. If not, it must be that $f_1(\mathbf{v}) \geq v_1$, $f_2(\mathbf{v}) \geq v_2$ and $f_3(\mathbf{v}) \geq v_3$. However, as $v_1 + v_2 + v_3 = 1 = f_1(\mathbf{v}) + f_2(\mathbf{v}) + f_3(\mathbf{v})$, we have that $f_i(\mathbf{v}) = v_i$ for $i = 1, 2, 3$ so $f(\mathbf{v}) = \mathbf{v}$ or equivalently, $\mathbf{v}$ is a fixed point, a contradiction.[96]

Next, let us show that the colouring is in fact a Sperner colouring. First we need that $\mathbf{x}_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is red, $\mathbf{x}_b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is blue and $\mathbf{x}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is green.[97] Next, for any vector on the line segment $\mathbf{x}_a$ and $\mathbf{x}_b$ we have to show that it is either red or blue. Well every vector on this segment has a zero for the third component, so it can not have the colour green. As such, it must be either red or blue as desired. Similarly, we can show that any vector on the segment $(\mathbf{x}_b, \mathbf{x}_c)$ is blue or green and any vector on the segment $(\mathbf{x}_a, \mathbf{x}_c)$ is red or green.

[95] Remember points in Δ^2 are 3-dimensional non-negative vectors whose components add up to one.

[96] Show this, i.e. if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $x_i \leq y_i$ for all i then $x_i = y_i$ for all i.

[97] Show this.

Given this Sperner colouring, we know that there must be a small triangle with three different colours. Let T_i be this small triangle and let $\mathbf{y}, \mathbf{z}, \mathbf{w}$ be the vertices of this triangle with colours, red, blue, green respectively. We know that $y_1 > f_1(\mathbf{y})$, $z_2 > f_2(\mathbf{z})$ and $w_3 > f_3(\mathbf{w})$.

The idea is to take a sequence of finer and finer triangulations. Each time, we can find a smaller and smaller triangle with three different colours and vertices $\mathbf{y}$ in red, $\mathbf{z}$ in blue and $\mathbf{w}$ in green. This creates a sequence of vectors,

$$(\mathbf{y}_n)_{n\in\mathbb{N}}, (\mathbf{z}_n)_{n\in\mathbb{N}}, (\mathbf{w}_n)_{n\in\mathbb{N}}.$$

Where the vectors $\mathbf{y}_n, \mathbf{z}_n, \mathbf{w}_n$ get closer and closer together as $n \to \infty$.

Now, given that the space Δ^2 is compact, we can find subsequences that converge,[98]

$$\mathbf{y}_{\varphi(n)} \to \mathbf{y},\ \mathbf{z}_{\varphi(n)} \to \mathbf{z},\ \mathbf{w}_{\varphi(n)} \to \mathbf{w}.$$

[98] First take a subsequence along $\mathbf{y}_n$ converges. Then from this subsequence, take a further subsequence along which $\mathbf{z}_n$ also converges and from this subsubsequence, take a subsequence along also $\mathbf{w}_n$ converges.

Of course, given that the distances between these vectors $\mathbf{y}_{\varphi(n)}, \mathbf{z}_{\varphi(n)}, \mathbf{w}_{\varphi(n)}$ become smaller and smaller, it must be that $\mathbf{y} = \mathbf{z} = \mathbf{w}$, i.e. the subsequences converge to the same vector. Also, for all n,

$$\begin{aligned} f_1(\mathbf{y}_{\varphi(n)}) &< y_{\varphi(n),1}, \\ f_2(\mathbf{z}_{\varphi(n)}) &< z_{\varphi(n),2}, \\ f_3(\mathbf{w}_{\varphi(n)}) &< w_{\varphi(n),3}. \end{aligned}$$

Now, taking the limit for $\varphi(n) \to \infty$ from both left and right hand side, and given continuity of f, we get,

$$\begin{aligned} f_1(\mathbf{y}) &\le y_1, \\ f_2(\mathbf{y}) &\le y_2, \\ f_3(\mathbf{y}) &\le y_3. \end{aligned}$$

Also $\mathbf{y} \in \Delta^2$ so $y_1 + y_2 + y_3 = f_1(\mathbf{y}) + f_2(\mathbf{y}) + f_3(\mathbf{y})$ which gives that $f(\mathbf{y}) = \mathbf{y}$ or equivalently, $\mathbf{y}$ is a fixed point. This gives the desired contradiction. $\square$

Now, let's go back to Brouwer's theorem: if $f : S \to S$ is continuous and S is a compact and convex set, then f has a fixed point. We have shown that the theorem is true for $S = \Delta^n$, so how can we generalize this result for all S, compact and convex. We we can use the following lemma.

Theorem 23. *Let T be a compact set in $\mathbb{R}^n$ such that there is a one to one continuous function $g : T \to \Delta^n$ such that g^{-1} is also continuous. If $f : T \to T$ is continuous, then f has a fixed point.*

Proof. Consider the mapping $h : \Delta^n \to \Delta^n$ where $h(x) = g(f(g^{-1}(x)))$. So it takes a point $\mathbf{x}$ from Δ^n maps it to T via g^{-1}. Then maps it according to f and transfers it back to Δ^n via g.

The function h is a continuous map from S to S as it is the composition of continuous functions. Also, by the previous theorem h has a fixed point in Δ^n, i.e. a vector $\mathbf{x} \in \Delta^n$ such that $h(\mathbf{x}) = \mathbf{x}$. But then,

$$g(f(g^{-1}(\mathbf{x}))) = \mathbf{x},$$
$$\iff f(g^{-1}(\mathbf{x})) = g^{-1}(\mathbf{x}).$$

As such, $g^{-1}(\mathbf{x}) \in T$ is a fixed point of f. □

Given this result, the only thing we need to show is that if S is convex and compact then there exists a one-to-one and continuous function $g : S \to \Delta^n$ for which g^{-1} is also continuous. Luckily this is the case,[99]

[99] I skip the proof of this result.

Lemma 5. *If $S \subseteq \mathbb{R}^k$ is convex and compact then there is a $m \leq k$ and a one to one continuous function $g : S \to \Delta^m$ such that g^{-1} is also convex.*

Exercises

1. Let $f(x) = x^2$ and suppose that $S =]0, 1[$. Show that f has no fixed point even though it is a continuous mapping from S to S. Does this contradict Brouwer's theorem, why, why not?

2. Use Brouwer's theorem to show that the equation $\cos(x) - x - 1/2 = 0$ has a solution in the interval $0 \leq x \leq \frac{\pi}{4}$.

3. Place a map of your country on the floor. Show that there is at least one point on the map that is exactly above the point it is referring to.

4. Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded, then f has a fixed point.

5. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to have bounded displacement if there is a number M such that for all $\mathbf{x} \in \mathbb{R}^n$, $\|f(\mathbf{x}) - \mathbf{x}\| \leq M$. Show that any such continuous function is onto (i.e. maps to every point in $\mathbb{R}^n$).

General equilibrium in exchange economies

> When Debreu received his Nobel prize in 1983, he was accosted by journalists who wanted to know his views about where the economy was headed. He is said to have thought awhile and then continued, "Imagine an economy with n goods and m consumers. . . "

A SIZEABLE PART of economics looks at the optimal behaviour of consumers and producers buying and selling goods. These consumers and producers meet each other on markets. The transaction of goods is governed by (relative) prices. And as equilibrium dictates, prices should be set as to equalize demand and supply.

As economists we are find it obvious that there should be some prices at which markets clear. However, if you think more carefully about this, there is something peculiar going on. Prices for certain goods, will affect the demand and supply for other goods and therefore their prices. These price adjustments in turn affect demand and supply behaviour on other markets and therefore the prices of other goods. Given this, it is not at all obvious that there should exist a single price vector that equates the supply and demand for all goods simultaneously. The fact that such equilibrium price vector exists, is one of the main theoretical results of economic theory of the past century. Its proof relies heavily on Brouwer's fixed point theorem.

The proof is due to Lionel McKenzy, Kenneth Arrow and Gérard Debreu.

CONSIDER AN ECONOMY with N consumers. Each consumer has an endowment of goods. We denote the endowment of consumer i by $\mathbf{e}^i \in R^k_+$. The endowment is simply a vector of good quantities, e^i_j is the amount of good j that consumer i has to start with. Each consumer also has a utility function $u^i : \mathbb{R}^k \to \mathbb{R}$ where $u^i(\mathbf{q})$ gives the utility of consumer i for consuming the bundle $\mathbf{q} \in \mathbb{R}^k_+$. The economy can be represented by the tuple

$$\mathcal{E} = \{N, (u^i, \mathbf{e}^i)_{i \le N}\}.$$

We consider the case of an exchange economy. This means that there is no production (firms) involved. However, consumers are allowed to trade with each other.

We require that that the utility functions u^i are continuous and strictly quasi-concave. We also assume that goods are traded (i.e. bought and sold) according to some price vector $\mathbf{p} \in \mathbb{R}^k_{++}$. If every consumer is price taker, then the budget constraint of consumer i (i.e. the goods that she can consume) is given by,

$$B^i(\mathbf{p}) = \{\mathbf{q} \in \mathbb{R}^k_+ : \mathbf{p} \cdot \mathbf{q} \leq \mathbf{p} \cdot \mathbf{e}^i\}.$$

It can be shown that this correspondence $B^i(\mathbf{p})$ is continuous on $\mathbb{R}^k_{++}$. As such,

$$\max_{\mathbf{q} \in B^i(\mathbf{p})} u^i(\mathbf{q}),$$

leads, by Berge's theorem, to an optimal solution, say the demand function $d^i : \mathbb{R}^k_{++} \to \mathbb{R}^k$, which gives the optimal consumption bundle of consumer i, $d^i(\mathbf{p})$ for a given price vector $\mathbf{p}$. We know this function is continuous for $\mathbf{p} \in \mathbb{R}^k_{++}$.

In addition to continuity and strict quasi-concavity, we also assume that utility functions are locally non-satiated.

Definition 25 (locally non-satiation)**.** *The function $u^i : \mathbb{R}^k_+ \to \mathbb{R}$ is locally non-satiated if for all $\mathbf{x} \in \mathbb{R}^k_+$ and all $\varepsilon > 0$ there is a $\mathbf{y} \in \mathbb{R}^k_+$ such that,*

$$\|\mathbf{x} - \mathbf{y}\| < \varepsilon \text{ and } u^i(\mathbf{y}) > u^i(\mathbf{x}).$$

Local non-satiation states that at any bundle there is always another bundle arbitrarily close that gives a higher utility.

The following result shows that if utility functions are locally non-satiated, then the budget constraint will be binding.

Lemma 6. *If u^i is locally non-satiated then $\mathbf{p} \cdot d^i(\mathbf{p}) = \mathbf{p} \cdot \mathbf{e}^i$.*

Proof. Let $\mathbf{x} = d^i(\mathbf{p})$. We know that $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{e}^i$. Towards a contradiction, assume that $\mathbf{p} \cdot \mathbf{x} < \mathbf{p} \cdot \mathbf{e}^i$. Let $\varepsilon = \mathbf{p} \cdot \mathbf{e}^i - \mathbf{p} \cdot \mathbf{x} > 0$. Then for $0 < \|\mathbf{y} - \mathbf{x}\| < \varepsilon / \|\mathbf{p}\|$, we have that,[100]

$$\begin{aligned} \mathbf{p} \cdot \mathbf{y} &= \mathbf{p} \cdot \mathbf{x} + \mathbf{p} \cdot (\mathbf{y} - \mathbf{x}), \\ &= \mathbf{p} \cdot \mathbf{e}^i - \varepsilon + \mathbf{p} \cdot (\mathbf{y} - \mathbf{x}), \\ &\leq \mathbf{p} \cdot \mathbf{e}^i - \varepsilon + \|\mathbf{p}\| \|\mathbf{y} - \mathbf{x}\|, \\ &< \mathbf{p} \cdot \mathbf{e}^i. \end{aligned}$$

As such there is a $\delta > 0$ such that for all $y \in \mathbb{R}^k$ with $\|y - x\| < \delta$, $\mathbf{p} \cdot \mathbf{y} \leq \mathbf{p} \cdot \mathbf{e}^i$.[101] Also, by locally non-satiation there should be at least one such $\mathbf{y}$ for which $u^i(\mathbf{y}) > u^i(\mathbf{x})$, a contradiction with optimality of $\mathbf{x}$. □

[100] Observe the use of the Cauchy-Schwartz inequality.

[101] Set $\delta = \varepsilon / a$.

Let $d^i(\mathbf{p})$ be the demand function of consumer i at prices $\mathbf{p}$. In addition, consumer i has an endowment $\mathbf{e}^i$. The excess demand function of consumer i is defined as,

$$d^i(\mathbf{p}) - \mathbf{e}^i.$$

If for goods j, $d^i_j(\mathbf{p}) - e^i_j > 0$, then the consumer will buy this good on the market. If on the other hand, good j has $d^i_j(\mathbf{p}) - e^i_j < 0$, then the consumer will sell this good on the market. If we sum the excess demand functions of all consumers together, we obtain the so called aggregate excess demand function.

Definition 26 (Aggregate excess demand function)**.** *The (aggregate) excess demand function is defined as,*

$$z(\mathbf{p}) = \sum_{i=1}^{N} (d^i(\mathbf{p}) - \mathbf{e}^i)$$

The following gives some properties of the excess demand function.

Lemma 7. *If $z(\mathbf{p}) : \mathbb{R}^k_{++} \to \mathbb{R}^k$ is an excess demand function then:*

- $\mathbf{p} \cdot z(\mathbf{p}) = 0.$
- *for all $\lambda > 0$, $z(\lambda \mathbf{p}) = z(\mathbf{p})$.*
- *$z : \mathbb{R}^k_{++} \to \mathbb{R}^k$ is continuous.*

Proof. We have that:

$$\begin{aligned} \mathbf{p} \cdot z(\mathbf{p}) &= \sum_{i=1}^{N} \left(\mathbf{p} \cdot d^i(\mathbf{p}) - \mathbf{p} \cdot \mathbf{e}^i \right), \\ &= 0, \end{aligned}$$

by the fact that every consumers budget is balanced. Homogeneity follows from the fact that $d^i(\lambda \mathbf{p}) = d^i(\mathbf{p})$. Rescaling of the prices does not change the budget constraint so it does not influence the optimization problem. Finally z is continuous as it is the sum of continuous functions. □

The first property $\mathbf{p} \cdot z(\mathbf{p}) = 0$ is called **Walras' law**. It holds for every price vector and is a simple consequence from the fact that all consumers have a balanced budget.

The homogeneity of degree zero property, i.e. $z(\lambda \mathbf{p}) = z(\mathbf{p})$ allows us to focus on price vectors $\mathbf{p} \in \mathbb{R}^k_{++}$ for which $\sum_i p_i = 1$.[102] This allows us to define the domain of the excess demand function to be the $k-1$ dimensional simplex. So we can set $z : \Delta^{k-1} \to \mathbb{R}^k$.

[102] Set $\lambda = \frac{1}{\sum_i p_i}$.

We say that the price vector $\mathbf{p}$ is an equilibrium price if demand is equal to supply on all markets. In other words, the excess demand function is zero.[103]

[103] Or the excess demand can be negative if prices are zero. These are then goods that nobody really wants.

Definition 27. *Let $z(\mathbf{p})$ be an excess demand function. Then $\mathbf{p}^*$ is an equilibrium price vector if*

$$z(\mathbf{p}^*) \leq \mathbf{0}.$$

and if $z_j(\mathbf{p}^*) < 0$, then $p_j = 0$.

Theorem 24. *Let $z : \Delta^{k-1} \to \mathbb{R}^k$ be a continuous function that satisfies Walras' law ($\mathbf{p} \cdot z(\mathbf{p}) = 0$ for all $\mathbf{p} \in \Delta^{k-1}$). Then there exists an equilibrium price vector.*

Proof. Consider an auctioneer whose task is to produce an equilibrium on all markets. What will this auctioneer do. Well she will increase the prices of the goods whose excess demand is strictly positive.[104]

Let:

$$g_j(\mathbf{p}) = \max\{0, z_j(\mathbf{p})\}.$$

If $g_j(\mathbf{p}) > 0$ then demand is greater than supply, so the price of good j should go up. Say the updated price evolves according to:

$$p_j + g_j(\mathbf{p}).$$

Of course, the auctioneer should also make sure that the prices remain in the simplex Δ^{k-1}. This can be done by dividing this "new price" by the sum of all "new prices", giving the updated price of good j by,[105]

$$\frac{p_j + g_j(\mathbf{p})}{1 + \sum_{t=1}^{k} g_t(\mathbf{p})}.$$

Define the function $f : \Delta^{k-1} \to \Delta^{k-1}$ by:[106]

$$f_j(\mathbf{p}) = \frac{p_j + g_j(\mathbf{p})}{1 + \sum_{t=1}^{k} g_t(\mathbf{p})}.$$

The functions $f_j(\mathbf{p})$ are continuous.[107] Now define $f(\mathbf{p}) = \begin{bmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \\ \vdots \\ f_k(\mathbf{p}) \end{bmatrix}$ which is then a continuous function from Δ^{k-1} to Δ^{k-1}. By Brouwer's fixed point theorem, f has a fixed point, say $\mathbf{p}^*$. Then, for all goods j,

$$p_j^* = \frac{p_j^* + g_j(\mathbf{p}^*)}{1 + \sum_{t=1}^{k} g_t(\mathbf{p}^*)},$$

$$\leftrightarrow p_j \sum_{t=1}^{k} g_t(\mathbf{p}^*) = g_j(\mathbf{p}^*).$$

Let us multiply each side of this equality by $z_j(\mathbf{p}^*)$ and sum over all goods j.

$$\left(\sum_{t=1}^{k} g_t(\mathbf{p}^*)\right)\left(\sum_{j=1}^{k} p_j z_j(\mathbf{p}^*)\right) = \sum_{j=1}^{k} z_j(\mathbf{p}^*) g_j(\mathbf{p}^*).$$

[104] And lower the price of the goods if the excess demand is negative.

[105] Notice that $\sum_j p_j = 1$.

[106] Check that $f_j(\mathbf{p}) \geq 0$ and $\sum_j f_j(\mathbf{p}) = 1$.

[107] This follows from the fact that $g_j(\mathbf{p})$ is continuous.

The left hand side is equal to zero by Walras' law $(\mathbf{p} \cdot z(\mathbf{p})) = 0$. As such,

$$\sum_{j=1}^{k} z_j(\mathbf{p}^*) g_j(\mathbf{p}^*) = \sum_{j=1}^{k} z_j(\mathbf{p}^*) \max\{0, z_j(\mathbf{p}^*)\} = 0.$$

If for good j, $z_j(\mathbf{p}^*) \leq 0$ then the corresponding term is zero.[108] On the other hand if $z_j(\mathbf{p}^*) > 0$ then the term is strictly positive.[109] As such, this is a sum of non-negative numbers that should add up to zero. Conclude that all terms should equal zero, or equivalently, for all goods j, $z_j(\mathbf{p}^*) \leq 0$.

Now, if for some good j, $z_j(\mathbf{p}^*) < 0$ then, by Walras' law, as $\mathbf{p}^* \cdot z(\mathbf{p}^*) = 0$ it must be that $p_j^* = 0$, which shows the second part. □

[108] As $g_j(\mathbf{p}^*) = 0$, we have that $z_j(\mathbf{p}^*) g_j(\mathbf{p}^*) = 0$.

[109] As $g_j(\mathbf{p}^*) > 0$.

A CAREFUL READER may have noticed that in Theorem 24 we assumed continuity of $z(\mathbf{p})$ for all price vectors in Δ^{k-1}. This includes the price vectors which have zero components. On the other hand, in Lemma 7 we demonstrated continuity of $z(\mathbf{p})$ for all price vectors whose components are strictly positive, i.e. $\mathbf{p} \gg \mathbf{0}$. We could only proof continuity for $\mathbf{p} \gg \mathbf{0}$ because our assumptions are not strong enough to ensure that individual demand functions $d^i(\mathbf{p})$ are continuous when the prices for some goods are equal to zero.[110] It is possible to account for this discrepancy at the cost of introducing some additional assumptions but this makes the proof considerably more complicated. I refer to the handbook of Jehle and Reny (Advanced Microeconomic Theory, 2011) for the fine details.

[110] If prices of a good is zero, it might be that the demand of this good goes to infinity.

Kakutani's fixed point theorem

> Shizuo Kakutani was a Japanese mathematician. At the start of World War II, he was a visiting professor at the Institute for Advanced Study in Princeton. With the outbreak of war he was given the option of staying at the Institute or returning to Japan. He chose to return because he was concerned about his mother. So he was put on a Swedish ship which sailed across the Atlantic, down around the Cape, and up to Madagascar, or thereabouts, where he and other Japanese were traded for Americans aboard a ship from Japan. The trip across the Atlantic was long and hard. There was the constant fear of being torpedoed by the Germans. What, you may wonder, did Kakutani do. He proved theorems. Every day, he sat on deck and worked on his mathematics. Every night, he took his latest theorem, put it in a bottle and threw it overboard. Each one contained the instruction that if found it should be sent to the Institute in Princeton. To this day, not a single letter has been received.
>
> Stanley Eigen, Annals of Improbable Research

Brouwer's fixed point theorem is very powerful. Sometimes, however we would like to have a fixed point for correspondences and not just functions. Remember, a function associates with each point in a domain a unique point in the range. A correspondence, on the other hand, can correspond to any point in the domain, multiple points in the range.

Definition 28. *Let $G : S \twoheadrightarrow S$ be a correspondence. The vector $\mathbf{x} \in S$ is a fixed point of G if $\mathbf{x} \in G(\mathbf{x})$.*

Figure 11: A correspondence with a fixed point.

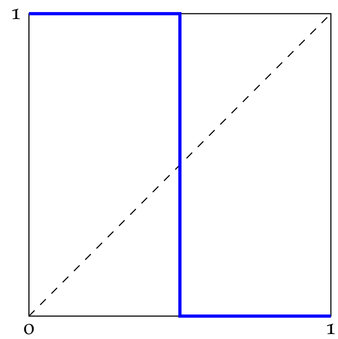


Figure 11 gives a correspondence with a fixed point: $1/2 \in G(1/2)$. This case, however, can not be handled by the Brouwer fixed point theorem as the correspondence is not a function.

The following gives the main fixed point results for correspondences, known as Kakutani's fixed point theorem.

Theorem 25 (Kakutani)**.** *Let $G : S \twoheadrightarrow S$ be a non-empty upper hemicontinuous convex valued correspondence on a compact and convex set S, then there is an $\mathbf{x} \in S$ such that $\mathbf{x} \in G(\mathbf{x})$.*

Proof. (sketch) The idea is to use Brouwer's fixed point after approximating the correspondence with a function. The problem is that there does not necessarily exist a continuous function $f : S \to S$ such that $f(\mathbf{x}) \in G(\mathbf{x})$ for all $\mathbf{x} \in S$. If such a function existed we could pick the fixed point of f, which would give $\mathbf{x} = f(\mathbf{x}) \in G(\mathbf{x})$, so we'd have a fixed point of G.

The idea is to slightly expand the graph of G by a small amount. In particular define the ε-ball around the graph of G,

$$B_\varepsilon(gr_G) = \left\{(\mathbf{x},\mathbf{y}) \in S \times S | \exists(\mathbf{x}',\mathbf{y}'), \mathbf{y}' \in G(\mathbf{x}'), \left\|(\mathbf{x},\mathbf{y}) - (\mathbf{x}',\mathbf{y}')\right\| < \varepsilon\right\}.$$

Define the correspondence G_ε where $\mathbf{y} \in G_\varepsilon(\mathbf{x})$ if $(\mathbf{x},\mathbf{y}) \in B_\varepsilon(gr_G)$. If the graph G is closed and convex, then for any $\varepsilon > 0$, there exists is a continuous function f_ε such that for all $\mathbf{x}$, $f_\varepsilon(\mathbf{x}) \in G_\varepsilon(\mathbf{x})$ (This is where the convexity part kicks in; this result is called von Neumann's approximation theorem.[111] By Brouwer's theorem, f_ε has a fixed point.

[111] We are skipping the proof.

In this way, we can find a sequence of continuous functions $(f_{1/n})_{n\in\mathbb{N}_0}$, such that $f_{1/n}(\mathbf{x}) \in G_{1/n}(\mathbf{x})$ for all $\mathbf{x} \in S$. Each of these functions has a fixed point, say $\hat{\mathbf{x}}_n$. Also, because of the way that the set $B_\varepsilon(gr_G)$ is defined, we know that for all n there are $(\mathbf{x}_n,\mathbf{y}_n)$ with $\mathbf{y}_n \in G(\mathbf{x}_n)$ such that $\|(\hat{\mathbf{x}}_n,\hat{\mathbf{x}}_n) - (\mathbf{x}_n,\mathbf{y}_n)\| < 1/n$.

As S is compact, the sequence $(\hat{\mathbf{x}}_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(\hat{\mathbf{x}}_{\varphi(n)})_{n\in\mathbb{N}}$ with $\mathbf{x}_{\varphi(n)} \overset{n}{\to} \hat{\mathbf{x}}$. But then $(\mathbf{x}_{\varphi(n)},\mathbf{y}_{\varphi(n)}) \overset{n}{\to} (\hat{\mathbf{x}},\hat{\mathbf{x}})$. As G is upper hemicontinuous, it follows that $(\hat{\mathbf{x}},\hat{\mathbf{x}})$ is in the graph of G, so $\hat{\mathbf{x}} \in G(\hat{\mathbf{x}})$ which gives us the fixed point. □

Exercises

1. Let C be a correspondence from $[0,2]$ into itself defined as

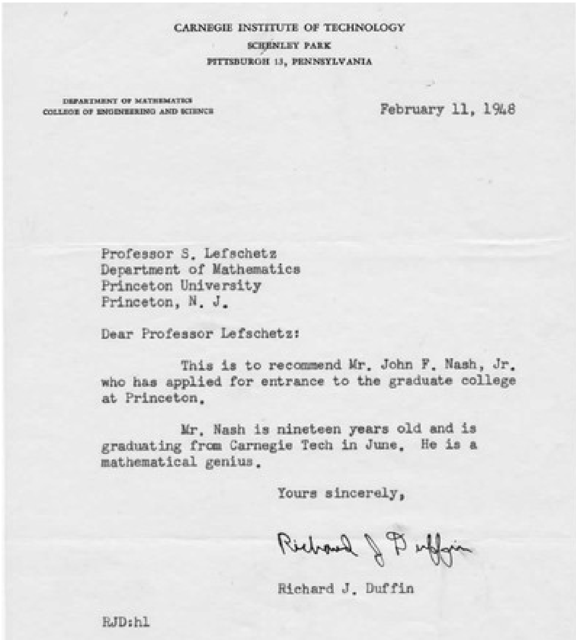
$$C(x) = \begin{cases} \{1\}, \text{ if } 0 \le x < 1, \\ [0,2] \text{ if } 1 \le x \le 2 \end{cases}$$

   Show that C satisfies all the conditions of Kakutani's theorem.

2. Verify whether the Kakutani fixed point theorem holds for the following correspondences from the unit interval $[0,1]$ to itself. For each correspondence, explain which conditions hold and fail, and identify the set of fixed points.

   - $G(x) = \{y \in X | \|y-x\| \le 1/5\}$.
   - $H(x) = \{y \in X | \|y-x\| \ge 1/10\}$.
   - $L(x) = \arg\max_y \|y-x\|$
   - $M(x) = co(L(x))$.

Existence Nash equilibrium



CARNEGIE INSTITUTE OF TECHNOLOGY
SCHENLEY PARK
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DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING AND SCIENCE

February 11, 1948

Professor S. Lefschetz
Department of Mathematics
Princeton University
Princeton, N. J.

Dear Professor Lefschetz:

This is to recommend Mr. John F. Nash, Jr. who has applied for entrance to the graduate college at Princeton.

Mr. Nash is nineteen years old and is graduating from Carnegie Tech in June. He is a mathematical genius.

Yours sincerely,

Richard J. Duffin

Richard J. Duffin

RJD:hl

(Recommendation letter by Richard Duffin, Nash's undergraduate advisor at the Carnegie Institute of Technology, to Solomon Lefschetz, a math professor at Princeton)

In this last chapter, we will use Kakutani's fixed point theorem to show a second great theoretical result in economics, namely the existence of a Nash equilibrium.

A normal form game is given by a tuple (N, u_i, A_i). Here, N is a finite set of players and $A_i \subseteq \mathbb{R}^k$ is the set of actions of player i. The sets A_i are assumed to be compact and convex. The functions $u^i : \prod_i A_i \to \mathbb{R}$ is the utility or payoff function of player $i \in N$. Each function u_i is assumed to be continuous and quasi-concave in own actions.

Let us restrict ourselves to a situation with only two players, 1 and 2.[112] The Nash equilibrium assumes that each player chooses her best action give the action chosen by the other player. This gives the

[112] The proof can easily be extended to more than two players at the cost of more confusing notation.

following optimal value and best response correspondences,

$$\begin{aligned} v_1(\mathbf{a}_2) &= \max_{\mathbf{a}_1 \in A_1} u_1(\mathbf{a}_1, \mathbf{a}_2), \\ B_1(\mathbf{a}_2) &= \{\mathbf{a}_1 \in A_1 : u_1(\mathbf{a}_1, \mathbf{a}_2) = v_1(\mathbf{a}_2)\}, \\ v_2(\mathbf{a}_1) &= \max_{\mathbf{a}_2 \in A_2} u_2(\mathbf{a}_1, \mathbf{a}_2), \\ B_2(\mathbf{a}_1) &= \{\mathbf{a}_2 \in A_2 : u_2(\mathbf{a}_1, \mathbf{a}_2) = v_2(\mathbf{a}_1)\}. \end{aligned}$$

By assumption, the sets A_1, A_2 are compact and convex and the utility functions are continuous and quasi-concave. Given this, we know that the problems are well defined and in particular $B_1(\mathbf{a}_1)$ and $B_2(\mathbf{a}_2)$ are upper hemi-continuous and convex valued.

A strategy profile is a Nash equilibrium if it is a best response for all players simultaneously.

Definition 29. *A strategy profile $(\mathbf{a}_1, \mathbf{a}_2)$ is a Nash equilibrium if $\mathbf{a}_1 \in B_1(\mathbf{a}_2)$ and $\mathbf{a}_2 \in B_2(\mathbf{a}_1)$.*

The idea is to reformulate the property of a Nash equilibrium as the fixed point of a correspondence. We do this in the following way. Let

$$B(\mathbf{a}_1, \mathbf{a}_2) = \{(\mathbf{b}_1, \mathbf{b}_2) \in A_1 \times A_2 : \mathbf{b}_1 \in B(\mathbf{a}_2), \mathbf{b}_2 \in B(\mathbf{a}_1)\}.$$

This is a correspondence from $A_1 \times A_2$ to $A_1 \times A_2$. We have that $(\mathbf{b}_1, \mathbf{b}_2) \in B(\mathbf{a}_1, \mathbf{a}_2)$ if $\mathbf{b}_1$ is a best response of player 1 to $\mathbf{a}_2$ and if $\mathbf{b}_2$ is a best response of player 2 to $\mathbf{a}_1$.

The crucial property of this correspondence is that $(\mathbf{a}_1, \mathbf{a}_2) \in B(\mathbf{a}_1, \mathbf{a}_2)$ if and only if $(\mathbf{a}_1, \mathbf{a}_2)$ is a Nash equilibrium. In other words, the Nash equilibria are the fixed points of the correspondence B.

Theorem 26. *Every normal form game has a Nash equilibrium.*

Proof. It is easy to check that $B : A_1 \times A_2 \twoheadrightarrow A_1 \times A_2$ is upper hemicontinuous and convex valued.[113]

113 This follows from the fact that both B_1 and B_2 are upper hemicontinuous and convex valued.

Also $A_1 \times A_2$ is convex and compact. As such, if we apply Kakutani's fixed point theorem, we find that there is a strategy pair $(\mathbf{a}_1, \mathbf{a}_2) \in B(\mathbf{a}_1, \mathbf{a}_2)$, i.e. a Nash equilibrium. $\square$

Exercises

1. Consider two firms in a Cournot duopoly. Assume that q_1 and q_2 are the output levels of firm 1 and 2 respectively. The inverse demand function is given by $P(q_1 + q_2)$.[114] Assume that both firms have cost functions given by $c_1(q_1)$ and $c_2(q_2)$. What conditions guarantee that there exists a Nash equilibrium?

114 The function $P(q)$ gives the price at which the consumer demands an amount q.

2. Consider a two player game where each player has a finite number of pure strategies. Consider the mixed extension where each player can choose a probability distribution over her pure strategies. Use above theorem to show the existence of a mixed strategy Nash equilibrium.