THOMAS DEMUYNCK

DYNAMIC OPTIMIZATION

Contents

Introduction

In these notes we wil deal with the following class of problems,

$$\max_{(a_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, a_t),$$
$$\text{subject to } x_t \in X, a_t \in A,$$
$$a_t \in \Gamma(x_t),$$
$$x_{t+1} = r(x_t, a_t),$$
$$x_0 \text{ given.}$$

Such problems contain the following ingredients.

1. A set of states, denoted by X and a set of actions, denoted by A. A state is denoted by $x \in X$ and an action is denoted by $a \in A$.

2. A correspondence $\Gamma : X \to A$ that determines for each state $x \in X$, which actions $a \in \Gamma(x)$ can be taken by the decision maker when the current state is x.[1]

3. An instantaneous payoff function $F(x, a)$ that determines the immediate benefit of taking an action $a \in A$ when the state is $x \in X$.

4. A transition function $r : X \times A \to X$ where $r(x, a)$ gives the state $y = r(x, a) \in X$ in the next period given that the current state is $x \in X$ and the current action is $a \in A$.

5. A discount rate $\beta \in (0, 1)$ that determines the trade-off between future and current payoffs.

[1] Think $\Gamma(x)$ as a budget constraint where x is the vector of prices and income and a is the consumption bundle.

The problem is to determine the optimal (infinite) sequence of actions $a_0, a_1, \ldots$ that should be taken in order to optimize the infinite horizon discounted payoff function:

$$\sum_{t=0}^{\infty} \beta^t F(x_t, a_t),$$

Although this problem may look like a standard optimization problem, there is one key difference. Namely, the optimization problem requires us to find an infinite number of values $(a_t)_{t=0}^{\infty}$ rather than a finite number of values. As such, it is not certain that the usual approach to solve standard optimization problems can also be used to solve this problem.[2]

[2] By usual, we mean the act of setting up the Lagrangean and take the corresponding Kuhn-Tucker first order conditions.

Before we attack the problem in full force, let us start by considering an example. We will choose the Ramsey-Cass-Koopmans model which extended the famous Solow model by permitting elastic savings rates.[3] The Ramsey-Cass-Koopman model is a representative consumer model with endogenous capital formation. In this model, we have an economy where capital is the only input in the production process. The output for a given amount of capital k is determined by a production function:

[3] Ramsey, Frank P. (1928), "A Mathematical Theory of Saving," Economic Journal. 38: 543-559.
Cass, David, (1965), "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies. 32: 233-240.
Koopmans, T. C., (1965), "On the Concept of Optimal Economic Growth," The Economic Approach to Development Planning. Chicago: Rand McNally. pp. 225-287.

$$f(k) = Ak^{\alpha}.$$

Where $\alpha \in (0,1)$ is the output elasticity of capital. There is a representative household that chooses a sequence of consumption levels $(c_t)_{t=0}^{\infty}$. The period t payoff of choosing c_t gives an instantaneous payoff of

$$u(c_t) = \ln(c_t).$$

The problem faced by the representative household is to choose a sequence of consumption amounts $(c_t)_{t=0}^{\infty}$ and a corresponding sequence of capital holdings $(k_t)_{t=0}^{\infty}$ to maximize discounted lifetime utility,

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t).$$

where $\beta \in (0,1)$ is an exogenous discount rate. The law of motion for the capital stock is given by:

$$k_{t+1} = Ak_t^{\alpha} - c_t.$$

Here k_{t+1} is the stock of capital in period $t+1$. It is equal to the total amount of output, $f(k_t) = Ak_t^{\alpha}$, minus the part of output that is used for immediate consumption, c_t. This law of motion gives a clear trade off. Consumption increases the payoff but decreases future consumption by lowering the next period's amount of capital. The final piece of information to set up the model is a fixed initial level of capital k_0. Combining all pieces, we obtain the following problem,

$$\begin{aligned}
&\max_{(c_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t),\\
&\text{s.t. } k_{t+1} = Ak_t^{\alpha} - c_t,\\
&\quad k_t, c_t \geq 0,\\
&\quad k_0 \text{ given.}
\end{aligned}$$

Translating this into the dynamic optimization framework from the beginning of the chapter, we obtain the following ingredients.

1. The state space X is given by the possible amounts of capital, $(= \mathbb{R}_+)$. A state is given by a stock of capital $k \in X$. The action space, A, is the possible set of consumption levels $(= \mathbb{R}_+)$. An action is an amount of consumption $c \in A$.

2. The correspondence $\Gamma(k)$ determines the possible consumption levels when the level of capital is equal to k. It is determined by,

$$\Gamma(k) = \{c \in \mathbb{R}_+ : c \leq Ak^{\alpha}\}.$$

3. The instantaneous payoff function is given by, $F(k, c) = \ln(c)$. In this setting, it is independent of the state k (for given c).

4. The transition function $r(k, c)$ that determines the next periods amount of capital is given by $r(k, c) = Ak^{\alpha} - c$.

5. The discount rate is given by β.

It is instructive to first solve this problem when the time horizon is finite instead of infinite. Let T be the final period. If $T = 0$, we obtain a static optimization problem whose solution depends on the initial capital stock k_0.

$$v_0(k_0) = \max_{c_0} \ln(c_0) \text{ s.t. } k_1 = Ak_0^{\alpha} - c_0; k_1, c_0 \geq 0.$$

Given that $k_1 \geq 0$ and $\ln(.)$ is strictly increasing, the optimal solution is to set $k_1 = 0$ and $c_0 = Ak_0^{\alpha}$.[4] The function $v_0(k_0)$ is called the value function. It only depends only on the initial capital stock as all future capital stocks are determined by the optimal choice of the consumption levels. Substituting $k_1 = 0$ and $c_1 = Ak_0^{\alpha}$ into the problem gives,

[4] As positive amounts of k_1 generate no additional utility, it is optimal to leave no money on the table after the final period.

$$v_0(k_0) = \ln(Ak_0^{\alpha}) = \ln(A) + \alpha \ln(k_0).$$

Now, let look at the problem when the final time period $T = 1$. In this case, we need to choose two consumption levels c_0 and c_1 and we obtain the problem:

$$\begin{aligned} v_1(k_0) = \max_{c_0,c_1}&\{\ln(c_0) + \beta\ln(c_1)\}, \\ \text{s.t. } k_1 &= Ak_0^\alpha - c_0, \\ k_2 &= Ak_1^\alpha - c_1, \\ c_0&, c_1, k_1, k_2 \geq 0. \end{aligned}$$

Given that $k_2 \geq 0$, one clearly sees that $k_2 = 0$ should hold at the optimum.[5] Given this, we can substitute the constraints $c_1 = Ak_1^\alpha$ and $c_0 = Ak_0^\alpha - k_1$ into the objective function.

[5] Again, there should be no money left on the table.

$$v_1(k_0) = \max_{k_1}\{\ln(Ak_0^\alpha - k_1) + \beta\ln(Ak_1^\alpha)\}.$$

The first order condition gives,

$$\begin{aligned} &-\frac{1}{Ak_0^\alpha - k_1} + \beta\alpha\frac{Ak_1^{\alpha-1}}{Ak_1^\alpha} = 0, \\ \to &-\frac{1}{Ak_0^\alpha - k_1} + \beta\alpha\frac{1}{k_1} = 0, \\ \to &k_1 = \frac{\alpha\beta}{1+\alpha\beta}Ak_0^\alpha. \end{aligned}$$

The last line gives the optimal solution for k_1. Plugging this solution back into the objective function gives the value of $v_1(k_0)$.

Observe that here $Ak_0^\alpha - k_1 = \frac{1}{1+\alpha\beta}Ak_0^\alpha = c_0 \geq 0$, so the constraints $c_0, c_1, k_1 \geq 0$ are satisfied. Additionally, it is easily verified that the objective function is strictly concave in k_1, so the solution characterized by the first order conditions is a global maximum.

$$\begin{aligned} v_1(k_0) &= \ln\left(Ak_0^\alpha - \frac{\alpha\beta}{1+\alpha\beta}Ak_0^\alpha\right) + \beta\ln\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}Ak_0^\alpha\right)^\alpha\right), \\ &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \alpha\ln(k_0) + \beta\ln\left(\frac{A^{1+\alpha}(\alpha\beta)^\alpha}{(1+\alpha\beta)^\alpha}\right) + \alpha^2\beta\ln(k_0), \\ &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta\ln\left(\frac{A^{1+\alpha}(\alpha\beta)^\alpha}{(1+\alpha\beta)^\alpha}\right) + \alpha(1+\alpha\beta)\ln(k_0), \end{aligned}$$

So far so good. extending the final period once more, we set $T = 2$. Then we can write the problem as,[6]

[6] In this case, we can set $k_2 = 0$ and substitute the constraints into the objective function.

$$v_2(k_0) = \max_{k_1,k_2}\{\ln(Ak_0^\alpha - k_1) + \beta\ln(Ak_1^\alpha - k_2) + \beta^2\ln(Ak_2^\alpha)\}.$$

The two first order conditions are,

$$\begin{aligned} \frac{1}{Ak_0^\alpha - k_1} &= \frac{\alpha\beta Ak_1^{\alpha-1}}{Ak_1^\alpha - k_2}, \\ \frac{1}{Ak_1^\alpha - k_2} &= \frac{\alpha\beta Ak_2^{\alpha-1}}{Ak_2^\alpha}. \end{aligned}$$

The solution is,[7]

[7] It is readily verified that this implies $c_0, c_1, c_2, k_1, k_2 \geq 0$. Also the objective function is strictly concave in (k_1, k_2) so the first order conditions are sufficient for a global maximum.

$$\begin{aligned} k_1 &= \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}Ak_0^\alpha, \\ k_2 &= \frac{\alpha\beta}{1+\alpha\beta}Ak_1^\alpha. \end{aligned}$$

Substituting these solutions into the objective function gives the value function $v_2(k_0)$. This expression is big mess.[8] We can iterate this procedure, and solve the problem for $T = 3, 4, 5, \ldots$. Doing this, it can be shown that the solution converges for $T \to \infty$ to the values,

[8] Try it.

$$\begin{aligned}
&v_\infty(k_0) = a + b\ln(k_0), \text{ with,}\\
&a = \frac{1}{1-\beta}\left[\ln(A(1-\alpha\beta)) + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right],\\
&b = \frac{\alpha}{1-\alpha\beta}.
\end{aligned}$$

This motivate the following procedure to solve the infinite horizon maximization problem: repeatedly solve the dynamic optimization problem for T finite, i.e. $T = 0, 1, 2, 3, \ldots$, and look whether the solution converges when considering $T \to \infty$.

There are several problems with this approach. First of all, it is is not sure whether we will always get a clean functional form for $v_t(k_0)$. In our special setting where $f(k) = Ak^\alpha$ and $u(c) = \ln(c)$, we did have a closed form expression, but this is not the case in general. If we don't have a closed form solution for $v_t(k_0)$ is not clear how we should proceed. Second, even if we obtain a closed form solution, the method is rather cumbersome. We need to solve the optimization problem for various time periods in order to see some convergence going on. Third, even assuming that we are able to solve the problem for several finite time periods, it is not certain that these solutions convergence to some limiting solution. Let alone that we are able to proof such convergence. Fourth, even if this convergence happens, nothing guarantees us that the limit of the finite horizon optimization problem also provides a solution for the infinite horizon problem. Finally, we have no idea that this limit solution is the unique solution.

Given the large number of unresolved issues, it might be a good idea to have a fresh look at the initial problem.

$$v(k_0) = \max_{(c_t)_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \ln(c_t) \text{ s.t. } c_t + k_{t+1} \leq Ak_t^\alpha,$$

Suppose that at time $t = 1$ we are in state $k_1 = k^*$. What, then, is the optimal choice of c_1. In order to solve this problem, we need to solve:

$$\begin{aligned}
&\max_{(c_t)_{t=1}^\infty} \sum_{t=1}^\infty \beta^t \ln(c_t) \text{ s.t. } c_t + k_{t+1} \leq Ak_t^\alpha, k_1 = k^*,\\
&= \beta \max_{(c_t)_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \ln(c_t) \text{s.t. } c_t + k_{t+1} \leq Ak_t^\alpha, k_0 = k^*,\\
&= \beta v(k^*).
\end{aligned}$$

This means that we can replace our orriginal problem by solving:

$$v(k_0) = \max_{c_0} \left\{ \beta^0 \ln(c_0) + \beta v(k^*) \right\} \text{ s.t. } k^* = Ak_0^\alpha - c_0,$$
$$= \max_{c_0} \left\{ \ln(c_0) + \beta v(Ak_0^\alpha - c_0) \right\}.$$

This shows that we can reformulate the infinite horizon problem as a recursive problem. The optimal value $v(k_0)$ for an initial capital stock k_0 is determined by choosing c_0 to maximize current payoff $\ln(c_0)$ and the value of the future payoff which is conveniently written down as $\beta v(k_1) = \beta v(Ak_0^\alpha - c_0)$. The functional equation

$$v(k) = \max_{c \leq Ak^\alpha} \{\ln(c) + \beta v(Ak^\alpha - c)\}.$$

is called the **Bellman equation** of the dynamic optimization problem.[9] If we could somehow find out the value of the function $v(.)$, we could simply insert it into the right hand side, maximize this right hand side with respect to c and find out the optimal value for c for any initial level of capital k.

[9] A functional equation is an equation of where the unknown is an entire function instead of a single variable.

One way to find out $v(.)$ is to make an educated guess. Before, we found that the limiting value of the value function of the finite horizon problem was of the form $v(k) = a + b\ln(k)$. Substituting this into the Bellman equation gives,

$$a + b\ln(k) = \max_{c} \{\ln(c) + a\beta + b\beta \ln(Ak^\alpha - c)\},$$

The maximization problem on the right hand side gives the following first order conditions,[10]

[10] We see that the right hand side is concave in c and $c \geq 0$ so the first order conditions give a global maximum.

$$\frac{1}{c} - \frac{\beta b}{Ak^\alpha - c} = 0,$$
$$\rightarrow c = \frac{Ak^\alpha}{1 + \beta b},$$
$$\rightarrow Ak^\alpha - c = \frac{\beta b}{1 + \beta b} Ak^\alpha.$$

Plugging this into the Bellman equation gives,

$$a + b\ln(k) = \ln\left(\frac{Ak^\alpha}{1 + \beta b}\right) + a\beta + b\beta \ln\left(\frac{\beta b}{1 + \beta b} Ak^\alpha\right),$$

Matching up the coefficients on $\ln(k)$ gives,

$$b = \alpha(1 + \beta b),$$
$$\rightarrow b = \frac{\alpha}{1 - \alpha\beta}.$$

Matching up the constants gives,

$$a = \ln\left(\frac{A}{1 + \beta b}\right) + \beta a + \beta b \ln\left(\frac{\beta b A}{1 + \beta b}\right).$$

Substituting for b and solving for a finally gives,

$$a = \frac{1}{1-\beta}\left[\ln(A(1-\alpha\beta)) + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right],$$

This gives the same solution as before. In this case, we do have a closed form solution for the value function and for every initial capital stock k we know the optimal consumption level $c = \frac{Ak^\alpha}{1+\beta b}$. As such, the optimal solution can be found iteratively,

$$\begin{aligned}
c_0 &= \frac{Ak_0^\alpha}{1+\beta b}, k_1 = Ak_0^\alpha - c_0, \\
c_1 &= \frac{Ak_1^\alpha}{1+\beta b}, k_2 = Ak_1^\alpha - c_0, \\
&\ldots, \\
c_t &= \frac{Ak_t^\alpha}{1+\beta_b}, k_{t+1} = Ak_t^\alpha - c_t, \\
&\ldots
\end{aligned}$$

The tricky part of this approach is, obviously, that we have to guess the functional form of the value function $v(.)$ and there are only a few very specific instances where we can make a good guess about this functional form.

What then should we do if we don't know the form of the value function. Let's go back to the Bellman equation.

$$v(k) = \max_{c \le f(k)} \{u(c) + \beta v(Ak^\alpha - c)\}.$$

Can we still somehow use this equation to solve our problem. The answer is yes and the key to the solution lies in the 'recursiveness' of the equation.

Assume that we start with an "arbitrary" guess for the function $v(.)$, say $v_0(.)$. We know that v_0 does not satisfy the Bellman equation, but let us substitute it into the right hand side anyway. Doing this gives us on the left hand side a new function, say $v_1(.)$.[11]

$$v_1(k) = \max_c \{u(c) + \beta v_0(Ak^\alpha - c)\}.$$

Now, we can do the same thing with $v_1(.)$: plug it into the right hand side of the Bellman equation and look at the values that it generates on the left hand, giving us a new function $v_2(.)$.

$$v_2(k) = \max_c \{u(c) + \beta v_1(Ak^\alpha - c)\}.$$

[11] Observe that we start with a function $v_0(.)$ and get an entire new function $v_1(.)$ out of this by varying the level of k on the left and right hand side.

We can continue this process indefinitely, and generate functions $v_1(.), v_2(.), v_3(.), \ldots, v_n(.), \ldots$ What happens if we allow $n \to \infty$. We would hope that finally the function $v_n(.)$ converges to some limiting function v_∞ that satisfies our Bellman equation,

$$v_\infty(k) = \max_{c \leq f(k)} \{u(c) + \beta v_\infty(Ak^\alpha - c)\}.$$

This is the function we were looking for all along. Of course, currently, we don't know whether this iteration will converge to something useful or even that different starting functions for $v_0(.)$ will converge to the same limiting function $v_\infty(.)$. Studying the conditions for which this iteration does converge is the main objective of the theory developed in these notes.

Mathematical Preliminaries

In this chapter, we will introduce the necessary mathematical tools and results for the following chapters. We will need to have a look at the concepts of vector spaces and normed vector spaces. A special subclass of these spaces have the property that every Cauchy sequence has a limit, called Banach spaces.

Banach spaces will provide the necessary structure for our state space. We will define contraction mappings on these spaces and show that these have a unique fixed point. Additionally, we will present a useful result called Blackwell's theorem that gives an easy to verify set of conditions for a mapping to be a contraction mapping.

In a second part of the chapter, we will have a look at the theorem of the maximum. This celebrated result in economics gives us conditions for which the result of a maximization exists and satisfies some convenient continuity conditions.

Banach spaces

Before we can introduce the concept of a Banach space, we first need to define vector spaces.

Definition 1 (vector space)**.** *A* ***real vector space*** *X is a set of elements together with two operations, addition and scalar multiplication.*[12] *For any two vectors $x, y \in X$, addition gives a vector $x + y \in X$ and for any vector $x \in X$ and a real number $\alpha \in \mathbb{R}$, scalar multiplication gives $\alpha x \in X$. We have the following conditions on the operations of a vector space:*

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$
3. $\alpha(x + y) = \alpha x + \alpha y$,
4. $(\alpha + \beta)x = \alpha x + \beta x$,
5. $(\alpha\beta)x = \alpha(\beta x)$.

[12] The adjective real simply indicates that scalar multiplication is defined taking the reals, not elements of the complex plane or some other set.

6. $1x = x$.

Additionally, there is a zero element $\theta \in X$ such that,

7. $x + \theta = x$,

8. *for every $x \in X$ there is a $-x$ such that $x + (-x) = \theta$*

A first well known example of a vector space is the set of n-dimensional real vectors $\mathbb{R}^n$.[13] However, the concept of a vector space is much broader than vectors of numbers. We will mainly work with vector spaces that have real valued functions as elements. Consider two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ defined on some common domain D. Then we can define their sum $f + g$ as the function $h : D \to \mathbb{R}$ such that:

[13] Verify that this set satisfies all conditions.

$$h(x) \equiv (f + g)(x) = f(x) + g(x),$$

and the scalar product, αf ($\alpha \in \mathbb{R}$) as the function $h : D \to \mathbb{R}$ such that:

$$h(x) \equiv (\alpha f)(x) = \alpha f(x).$$

It is clear that these operations satisfy all eight conditions of a vector space.[14] As such, the set $F(D)$ of real valued functions on a common domain D forms a vector space. We can actually go further. If D is a topological space and if f and g are continuous functions from X to $\mathbb{R}$, then $f + g$ and αf are also continuous functions, so the set of all continuous real valued functions with domain D is also a vector space. Let us call this space $C(D)$.

[14] Here we define the null-vector θ to be the function $\theta(x) = 0$ for all $x \in X$.

The following are real vector spaces:

- The finite Euclidean space $\mathbb{R}^n$
- The set $X = \{x \in \mathbb{R}^2 : x = \alpha z\}$, where $z \in \mathbb{R}^2$
- The set of all continuous functions on $[a, b]$,

The following are not vector spaces

- The unit circle in $\mathbb{R}^2$ with the usual addition
- the set of all integers
- The set of non-negative functions on $[a, b]$.

We are now ready to define the notion of a norm on a vector space.

Definition 2 (normed vector space)**.** *A **norm** on a vector space X is a function $\|.\| : X \to \mathbb{R}_+$ such that for all $x, y \in X$ and $\alpha \in \mathbb{R}$,*

- $\|x\| \geq 0$, *with equality if and only if $x = \theta$,*
- $\|\alpha x\| = |\alpha| \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X together with a norm $\|.\|$ is called a normed vector space. Intuitively, the idea is that $\|x - y\|$ measures the distance between x and y. In particular, $\|x\|$ measures the distance between the zero element θ and x. The last condition is called the triangle inequality. Substituting x by $x - z$ and y by $z - y$ gives:

$$\|x - y\| \leq \|x - z\| + \|z - y\|.$$

In other words, the distance between x and y is always smaller than the distance between x and z plus the distance from z to y.

The following are normed vector spaces:

- $\mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
- $\mathbb{R}^n$ with $\|x\| = \max_i |x_i|$,
- $\mathbb{R}^n$ with $\|x\| = \sum_{i=1}^n |x_i|$.
- The set of all bonded infinite sequences $(x_1, \ldots)$ with $\|x\| = \sup_k |x_k|$ this space is called ℓ_∞.
- The set of continuous functions on $[a, b]$ with $\|x\| = \sup_{a \leq t \leq b} |x(t)|$ this space is called $C[a, b]$.
- The set of continuous functions on $[a, b]$ with $\|x\| = \int_a^b |x(t)| dt$.

Now, consider our previously defined vector space $C(D)$ of continuous real valued functions on a common domain D. Let us further restrict ourselves to the functions that are also bounded.[15] Let us call this the set $B(D)$. What would be a good norm on this set. In other words, if we take two bounded and continuous functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, how can we measure the 'distance' $\|f - g\|$ between these two functions?

[15] A function $f : D \to \mathbb{R}$ is bounded if there exists a number $M > 0$ such that for all $x \in D$, $|f(x)| \leq M$. Observe that M is chosen independent of x. Notice that if D is bounded, then any continuous function $f \in C(D)$ is also bounded, so in this case $C(D) = B(D)$.

A first idea would be to take one particular value $x_0 \in X$ and to define,

$$\|f - g\| = |f(x_0) - g(x_0)|.$$

In particular $\|f\| = |f(x_0)|$. The problem with this 'norm', however, is that it does not satisfy the first condition: it is possible that $|f(x_0)| = 0$, i.e. $f(x_0) = 0$, but f is not equal to the zero function. This can be fixed by taking the maximal distance between f and g over the set D,

$$\|f - g\| = \max_{x \in D} |f(x) - g(x)| \text{ in particular } \|f\| = \max_{x \in D} |f(x)|.$$

The problem with this proposal is that the maximum may not exist (if for example D is not compact). We can solve this by taking the supremum instead of the maximum.[16]

[16] The sup exists because we assumed that both f and g are bounded.

$$\|f - g\| = \sup_{x \in D} |f(x) - g(x)| \text{ in particular } \|f\| = \sup_{x \in D} |f(x)|.$$

This metric is called the sup or infinity norm.[17]

[17] Show that it satisfies all three conditions to be a norm.

For the analysis in the next chapters, it will be useful to generalize the notion of the sup-norm. Let us go back to the set of continuous functions on the set D, which we denoted by $C(D)$. Let $\phi : D \to \mathbb{R}_{++}$ be a continuous function that takes only strictly positive values. For such given function ϕ, we consider the set of functions $B_\phi(D)$ for which the function $\frac{f(x)}{\phi(x)}$ is bounded. In other words, $f \in B_\phi(D)$ if f is continuous and there exists an M such that for all $x \in D$, $\frac{f(x)}{\phi(x)} \leq M$.

For these functions, we can consider the following norm,

$$\|f\|_\phi = \sup_{x \in D} \frac{|f(x)|}{\phi(x)}.$$

Let us first show that this is indeed a norm. First,

$$\|f\|_\phi \geq 0,$$

is easily established.[18] If $\|f\|_\phi = 0$. Then we have that for all $x \in D$,

[18] Indeed, both $|f(x)| \geq 0$ and $\phi(x) > 0$.

$$0 = \frac{|f(x)|}{\phi(x)}.$$

Given that $\phi(x) > 0$, we have indeed that $f(x) = 0$ for all $x \in D$, so f is the zero function. Next,

$$\|\alpha f\|_\phi = \sup_{x\in D} \frac{|\alpha f(x)|}{\phi(x)} = |\alpha| \sup_{x\in D} \frac{|f(x)|}{\phi(x)} = |\alpha|\|f\|_\phi.$$

finally,

$$\begin{aligned}\|f+g\|_\phi &= \sup_{x\in D} \frac{|f(x)+g(x)|}{\phi(x)}, \\ &\le \sup_{x\in D} \frac{|f(x)|+|g(x)|}{\phi(x)}, \\ &\le \sup_{x\in D} \frac{|f(x)|}{\phi(x)} + \sup_{x\in X} \frac{|g(x)|}{\phi(x)}, \\ &= \|f\|_\phi + \|g\|_\phi.\end{aligned}$$

Observe that if we consider the constant function $\phi(x) = 1$ for all $x \in D$, then $\|f\|_\phi = \|f\|$. As such, the sup norm is a special case of the ϕ-norm with $\phi(x) = 1$ for all x. However, the ϕ-norm covers other cases to. Consider, for example $X = \mathbb{R}$ and $\phi(x) = |x| + 1$ then we see that f is bounded in the norm $\|.\|_\phi$, if f does not grow faster than $|x|$.[19] In other words, f can be unbounded but not 'more' unbounded than the function $\phi(x) = |x| + 1$. In general f will be bounded in the ϕ-norm if the value of $|f(x)|$ does not 'grow' faster than $\phi(x)$.

[19] For example, $f(x) = x^2$ is not bounded using this norm on $\mathbb{R}$. but $f(x) = ax + b$ is bounded although $f(x) = ax + b$ is not bounded in the sup-norm.

A MAIN REASON for introducing norms is to measure distance between different elements of a vector space. Once we can measure distances, we can also start talking about convergence.

Definition 3 (convergence)**.** *Let $(X, \|.\|)$ be a normed vector space. A countable sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X is said to **converge** to an element $x \in X$ if for all $\varepsilon > 0$, there exists a N_ε such that for all $n \ge N_\varepsilon$,*[20]

$$\|x_n - x\| < \varepsilon.$$

We also write this as $x_n \xrightarrow{n} x$ or $\lim_{n\to\infty} x_n = x$.

[20] We write N_ε to make it clear that N_ε may be different for different values of ε.

In words, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to an element x if for all strictly positive numbers ε, it is possible to go far enough in the sequence, say further than the N_ε'th element such that for all elements x_n beyond this element the distance between x_n and x is smaller than ε.[21]

[21] Alternatively, you could say that for any strictly positive number ε there are only a finite number of elements in the sequence $(x_n)_{n\in\mathbb{N}}$ that are at a distance greater than ε from x.

Next, we need the definition of a Cauchy sequence.

Definition 4 (Cauchy sequence)**.** *Let $(X, \|.\|)$ be a normed vector space. A sequence $(x_n)_{n\in\mathbb{N}}$ in S is a **Cauchy sequence** if for all $\varepsilon > 0$, there is a number N_ε such that for all $n, m \ge N_\varepsilon$,*

$$\|x_n - x_m\| < \varepsilon.$$

So a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if for any strictly positive number ε it is possible to go far enough in the sequence, further than N_ε such that the distance between any two elements beyond the N_ε'th position is less than ε.

Exercises:

- Show that if $x_n \to x$ and $x_n \to y$ then $x = y$.
- Show that if a sequence is convergent, then it satisfies the Cauchy criterion.
- Show that $x_n \to x$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

Complete metric spaces

It is always the case that a convergent sequence $x_n \xrightarrow{n} x$ in a normed vector space is also a Cauchy sequence. The reverse, however is not necessarily the case. In other words, it is possible that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, but it does not converge to an element in X. Normed vector spaces where this is true are called complete vector spaces.

Definition 5 (complete metric spaces)**.** *A normed vector space $(X, \|.\|)$ is complete if every Cauchy sequence in X converges to an element in X.*

Not every vector space is complete. As an example, consider the set $C([0,1])$ of continuous functions on the closed interval $[0,1]$ and consider the L^2 norm:

$$\|f\| = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}.$$

Let f_n be the step function that is equal to 0 on the interval $[0, 1-2^{-n}]$ and linearly rises to 1 on the interval $[1-2^{-n}, 1]$. One can show that this is a Cauchy sequence. However, its limit is not a continuous function.

Intuitively, a complete vector space is a space without any 'points' missing, where the missing points could either lie inside or at its boundary. We take it as a fact that the set of real numbers $\mathbb{R}$ with the norm $|x-y|$ is a complete vector space.[22] A complete normed vector space is also called a **Banach space**. We will use the term Banach space from now on.

[22] This is a consequence of the Bolzano-Weierstrass theorem.

The following theorem shows that $B_\phi(D)$ is a Banach space.

Theorem 1. *Let $\phi : D \to R_{++}$ be a continuous function and let $B_\phi(D)$ be the set of all continuous functions $f : D \to \mathbb{R}$ that are bounded in the norm $\|f\|_\phi = \sup_{x\in D} \frac{f(x)}{\phi(x)}$. Then $B_\phi(D)$ is a Banach space.*

Proof. That $B_\phi(D)$ is a normed vector space was shown above. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $B_\phi(D)$. We need to show that there exists an $f \in B_\phi(D)$ such that:

$$\lim_{n\to\infty} f_n = f \text{ or equivalently} \|f_n - f\|_\phi \xrightarrow{n} 0.$$

There are three steps. First, we find a candidate function f, second, we show that $(f_n)_{n\in\mathbb{N}}$ converges to this candidate function (in the $\|.\|_\phi$ norm). Third we show that the candidate function is in $B_\phi(D)$.

For step one, fix $x \in X$, then the sequence of real numbers $f_n(x)$ satisfies,

$$\begin{aligned}|f_n(x) - f_m(x)| &= \phi(x)\frac{|f_n(x) - f_m(x)|}{\phi(x)},\\ &\leq \phi(x)\sup_{y\in D}\frac{|f_n(y) - f_m(y)|}{\phi(y)},\\ &= \phi(x)\|f_n - f_m\|_\phi \overset{n,m}{\to} 0.\end{aligned}$$

As such, the sequence $(f_n(x))_{n\in\mathbb{N}}$ satisfies the Cauchy criterion. Notice that this is a sequence in $\mathbb{R}$. Also, as $\mathbb{R}$ is complete, the sequence $(f_n(x))_{n\in\mathbb{N}}$ has a limit, call it $f(x)$, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$.[23] Doing this for all $x \in D$, defines a function $f : D \to \mathbb{R}$ that we take to be our candidate function.

[23] This keeps x fixed and regards $(f_n(x))_{n\in\mathbb{N}}$ as a sequence of numbers in $\mathbb{R}$.

For step 2, we need to show that $\|f_n - f\|_\phi \overset{n}{\to} 0$. Let $\varepsilon > 0$ and let N be such that for $n, m \geq N$, $\|f_n - f_m\|_\phi < \varepsilon$. Then for all $n \geq N$

$$\begin{aligned}\frac{|f_n(x) - f(x)|}{\phi(x)} &= \frac{|f_n(x) - \lim_m f_m(x)|}{\phi(x)},\\ &= \lim_m \frac{|f_n(x) - f_m(x)|}{\phi(x)},\\ &\leq \lim_m \|f_n - f_m\|_\phi < \varepsilon.\end{aligned}$$

This holds for all x. As such, $\|f_n - f\|_\phi < \varepsilon$.

Finally, we need to show that $f \in B_\phi(D)$. Boundedness of $\|f\|_\phi$ is obvious.[24] Let us first show that $\frac{f(x)}{\phi(x)}$ is continuous. As $\phi(x)$ is continuous, this also shows that $f(x)$ is continuous. So, let us show that if $x_n \overset{n}{\to} x$ then $f(x_n)/\phi(x_n) \to f(x)/\phi(x)$.

[24] This follows from the fact that the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}\left|\frac{f(x_n)}{\phi(x_n)} - \frac{f(x)}{\phi(x)}\right| &= \left|\frac{f(x_n)}{\phi(x_n)} - \frac{f_m(x_n)}{\phi(x_n)} + \frac{f_m(x_n)}{\phi(x_n)} - \frac{f_m(x)}{\phi(x)} + \frac{f_m(x)}{\phi(x)} - \frac{f(x)}{\phi(x)}\right|,\\ &\leq \frac{|f(x_n) - f_m(x_n)|}{\phi(x_n)} + \left|\frac{f_m(x_n)}{\phi(x_n)} - \frac{f_m(x)}{\phi(x)}\right| + \frac{|f_m(x) - f(x)|}{\phi(x)},\\ &\leq \|f - f_m\|_\phi + \left|\frac{f_m(x_n)}{\phi(x_n)} - \frac{f_m(x)}{\phi(x)}\right| + \|f_m - f\|_\phi.\end{aligned}$$

The first and last term goes can be set arbitrarily small by picking m large enough as $(f_m)_{m\in\mathbb{N}}$ converges to f. The middle term goes to zero by continuity of f_m. □

When we take $\phi(x) = 1$ for all x, this theorem shows that $B(D)$ being the set of all continuous functions that are bounded in the sup norm $\|.\|$ norm[25] is also a Banach space.

[25] These are the continuous functions that are simply bounded.

Corollary 1. *Let $D \subseteq \mathbb{R}^n$ and let $B(D)$ be the set of all bounded continuous functions $f : X \to \mathbb{R}$ with the sup norm $\|f\| = \sup_{x \in D} |f(x)|$. Then $B(D)$ is a Banach space.*

Contraction mappings

Now that we are equipped with the notion of a Banach space, we can have a look at contraction mappings.

Definition 6 (contraction mapping)**.** *Let $(X, \|.\|)$ be a normed vector space and let $T : X \to X$ be a function mapping X into itself. The operator T is a **contraction mapping** with modulus $\beta \in [0, 1[$ if for all $x, y \in X$:*

$$\|T(x) - T(y)\| \leq \beta \|x - y\|.$$

A function is a contraction mapping if the distance between the two images of points x and y are closer together than the original points x and y. Intuitively, when we iterate such mapping, the points will at each step come closer and closer together. Eventually, we expect these iterations to converge to what we call a **fixed point**.

Definition 7. *Let $(X, \|.\|)$ be a normed vector space and let $T : X \to X$. Then $x \in X$ is called a fixed point of T if*

$$T(x) = x.$$

Let $B_\phi(D)$ be our set of continuous functions on D that are bounded in the ϕ-norm. A mapping T from $B_\phi(D)$ to $B_\phi(D)$ takes a function $f \in B_\phi(D)$ as input and produces another function $g = T(f) \in B_\phi(D)$. A fixed point of T is a function $f \in B_\phi(D)$ such that T maps f to itself: $f = T(f)$. A function T that takes functions to functions is called, for clarity, an operator. Often we omit the brackets when using operators, so we write Tf instead of $T(f)$. If we are interested in the value of $T(f)$ at a particular point $x \in D$, we can write this as $Tf(x)$ or sometimes (to avoid confusion) as $(Tf)(x)$.

Every contraction mapping (operator) on a Banach space has a unique fixed point.

Theorem 2 (Banach's contraction mapping theorem)**.** *Let $(X, \|.\|)$ be a Banach space and let $T : X \to X$ be a contraction mapping on X with modulus β, then*

- *T has exactly one fixed point $x \in X$,*
- *For any $x_0 \in X$, $\|(T^n x_0) - x\| \leq \beta^n \|x_0 - x\|$.*

Proof. Define the iterates of T, the mappings $(T^n)_{n\in\mathbb{N}}$ by $T^0x = x$, $T^nx = T(T^{n-1}x) = (T \circ T^{n-1})(x)$. Choose $x_0 \in S$ and let $(x_n)_{n\in\mathbb{N}}$ be defined as $x_n = T^nx_0$. By the contraction mapping property,

$$\|x_2 - x_1\| = \|Tx_1 - Tx_0\| \leq \beta\|x_1 - x_0\|.$$

By induction, we can show that,

$$\|x_{n+1} - x_n\| \leq \beta^n\|x_1 - x_0\|.$$

As such, for any $m \geq n$,

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \ldots + \|x_{n+1} - x_n\|,\\
&\leq \left[\beta^{m-1} + \ldots, +\beta^{n+1} + \beta^n\right]\|x_1 - x_0\|,\\
&= \beta^n\left[\beta^{m-n-1} + \ldots + \beta + 1\right]\|x_1 - x_0\|,\\
&\leq \frac{\beta^n}{1-\beta}\|x_1 - x_0\| \xrightarrow{n} 0.
\end{aligned}$$

This shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so it has a limit $x_n \xrightarrow{n} x \in X$. To show that $Tx = x$ note that

$$\begin{aligned}
\|Tx - x\| &\leq \|Tx - T^nx_0\| + \|T^nx_0 - x\|,\\
&\leq \beta\|x - T^{n-1}x_0\| + \|T^nx_0 - x\|,
\end{aligned}$$

Both terms on the right hand side converge to zero as $n \to \infty$ and the left hand side is independent of n. As such, $\|Tx - x\| = 0$, or equivalently, $Tx = x$. For uniqueness, assume that $x, \hat{x}$ are both fixed points of T, then

$$\|\hat{x} - x\| = \|T\hat{x} - Tx\| \leq \beta\|\hat{x} - x\|.$$

This can only be true if $\|\hat{x} - x\| = 0$ or $\hat{x} = x$. □

It will often be convenient to restrict the region in the set X where the fixed point is situated. Let $(X, \|.\|)$ be a Banach space and let X' be a closed subset of X. It can be shown that the smaller set $(X', \|.\|)$ is also a Banach space.[26] If $T : X \to X$ is a contraction mapping and if T maps X' to X' then T is also contraction mapping on the smaller set $(X', \|.\|)$.[27] As X' is closed, the unique fixed point of T will lie in X'. This is the gist of the following lemma.

[26] A set X' is closed if for all sequences $(x_n)_{n\in\mathbb{N}}$ in X', $x_n \xrightarrow{n} x$ (according to the norm $\|.\|$) implies that $x \in X'$.

[27] This requires that $T(X') \subseteq X'$.

Lemma 1. *Let $(X, \|.\|)$ be a Banach space and let $T : X \to X$ be a contraction mapping with fixed point $x \in X$. If X' is a closed subset of X and $T(X') \subseteq X'$, then $x \in X'$. If in addition $T(X') \subseteq X'' \subseteq X'$ then $x \in X''$.*

Proof. Choose $x_0 \in X'$ and note that $(T^n x_0)_{n\in\mathbb{N}}$ is a sequence in X' converging to the fixed point x of T. Since X' is closed, it follows that $x \in X'$, so the unique fixed point is also in X'. If in addition $T(X') \subseteq X''$ then it follows that $x = Tx \in X''$ so x is also in X''. □

The second part of the contraction mapping theorem provides a bound on the distance from the n-th iterate $T^n x_0$ to the fixed point x,

$$\|T^n x_0 - x\| \leq \beta^n \|x_0 - x\|.$$

This bound, however, is not very useful as it involves the 'unknown' limit x. The following gives a computationally more relevant bound.

Lemma 2. *Let $(X, \|.\|)$ be a Banach space, T a contraction mapping and x the fixed point of T. Then*

$$\|T^n x_0 - x\| \leq \frac{1}{1-\beta}\|T^n x_0 - T^{n+1} x_0\|.$$

Proof. Notice that,

$$\begin{aligned}\|T^n x_0 - x\| &\leq \|T^n x_0 - T^{n+1} x_0\| + \|T^{n+1} x_0 - x\|, \\ &\leq \|T^n x_0 - T^{n+1} x_0\| + \beta\|T^n x_0 - x\|.\end{aligned}$$

Rearranging this inequality gives the desired result. □

Previously, we saw that the set of continuous functions $f : D \to \mathbb{R}$ that are bounded in the $\|.\|_\phi$ norm, i.e. $B_\phi(D)$ was a Banach space. We will mainly be interested in contraction mappings from $B_\phi(D) \to B_\phi(D)$. These contraction mappings take functions in $B_\phi(D)$ to other functions in $B_\phi(D)$. The following theorem, known as Blackwell's theorem provides sufficient, easy to verify, conditions for an operator T to be a contraction mapping on $B_\phi(D)$.

Theorem 3 (Blackwell's sufficient conditions)**.** *Let $\phi : D \to R_{++}$ be a continuous function and let let $B_\phi(D)$ be the space of continuous functions $f : D \to \mathbb{R}$, that are bounded in the norm $\|.\|_\phi$. Let $T : B_\phi(D) \to B_\phi(D)$ be an operator satisfying,*

- *(monotonicity) If $f, g \in B_\phi(D)$ and $f(x) \leq g(x)$ for all $x \in D$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in D$.*
- *(discounting) for all $x \in D$, $f \in B_\phi(D)$ and $a \geq 0$, there is some $\beta \in (0, 1)$ such that,*

$$T(f + a\phi)(x) \leq (Tf)(x) + (\beta a)\phi(x),$$

Then T is a contraction with modulus β.

Proof. Observe that

$$\begin{aligned}\frac{f(x)}{\phi(x)} &= \frac{g(x)}{\phi(x)} + \frac{f(x)-g(x)}{\phi(x)},\\ &\le \frac{g(x)}{\phi(x)} + \frac{|f(x)-g(x)|}{\phi(x)},\\ &\le \frac{g(x)}{\phi(x)} + \|f-g\|_{\phi}.\end{aligned}$$

as such, multiplying both sides by $\phi(x) > 0$ gives,

$$f(x) \le g(x) + \phi(x)\|f-g\|_{\phi}.$$

So, by monotonicity

$$(Tf)(x) \le T(g + \|f-g\|_{\phi}\phi)(x) \le (Tg)(x) + \beta\|f-g\|_{\phi}\phi(x).$$

Equivalently,

$$\frac{(Tf)(x)-(Tg)(x)}{\phi(x)} \le \beta\|f-g\|_{\phi}.$$

Reversing the roles of f and g gives

$$\frac{(Tg)(x)-(Tf)(x)}{\phi(x)} \le \beta\|f-g\|_{\phi}.$$

This holds for all x and the right hand side does not depend on x, so taking the sup on the left hand side gives:

$$\|Tf - Tg\|_{\phi} \le \beta\|f-g\|_{\phi},$$

so T is a contraction mapping with modulus β. □

Applying above theorem to the case $\phi(x) = 1$,[28] we obtain the following, better known, version of Blackwell's theorem.

[28] This is the case where $\|.\|_{\phi}$ is the sup norm and $B(X)$ is the set of bounded continuous functions on X.

Corollary 2 (Blackwell's sufficient conditions). *Let $D \subseteq \mathbb{R}^l$ and let $B(D)$ be the space of bounded functions $f : D \to \mathbb{R}$, with the sup norm. Let $T : B(D) \to B(D)$ be an operator satisfying,*

- *(monotonicity) If $f, g \in B(D)$ and $f(x) \le g(x)$ for all $x \in D$, then $(Tf)(x) \le (Tg)(x)$ for all $x \in D$.*
- *(discounting) for all $x \in D$, there is some $\beta \in (0,1)$ such that,*

$$T(f+a)(x) \le (Tf)(x) + \beta a,$$

  for all $f \in B(D)$ and $a \ge 0$.

Then T is a contraction with modulus β.

Theorem of the maximum

In the second part of this chapter we'll have a look at a seminal result in mathematics, the theorem of the maximum. Consider two sets $X \subseteq \mathbb{R}^l$, and $A \subseteq \mathbb{R}^m$, and let $f : X \times A \to \mathbb{R}$ be a real valued function that takes a vector $x \in X$, a vector in $a \in A$ and produces a real number $f(x, a)$. Let $\Gamma : X \to A$ be a correspondence.[29] The theorem of the maximum deals with optimization problems of the following form,

$$\max_{a \in \Gamma(x)} f(x, a).$$

This problem optimizes a function $f(x, a)$ with respect to a, when a is restricted to lie in the set $\Gamma(x)$. Here x is kept fixed, so it is a parameter of the optimization problem. If for each x, $f(x, a)$ is continuous in a and the set $\Gamma(x)$ is nonempty and compact,[30] then for all x the maximum is attained.[31] In this case, we can define the function

$$v(x) = \max_{a \in \Gamma(x)} f(x, a),$$

and the correspondence:

$$G(x) = \{a \in \Gamma(x) : f(x, a) = v(x)\},$$

of values in $\Gamma(x)$ that attain this maximum. We would like to place additional restrictions such that the function v and the set G vary 'continuously' with the 'parameter' x.

Towards this end, we need to define the concepts of lower and upper hemi-continuity.

Definition 8 (Lower hemi-continuity). *The correspondence $\Gamma : X \to A$ is **lower hemi-continuous** (l.h.c.) at $x \in X$ if*

1. *$\Gamma(x)$ is non-empty*
2. *for every $a \in \Gamma(x)$ and every sequence $x_n \xrightarrow{n} x$, there exists an $N \geq 1$ and a sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \xrightarrow{n} a$ and $a_n \in \Gamma(x_n)$ for all $n \geq N$.*[32]

Definition 9 (Upper hemi-continuity). *A compact-valued correspondence $\Gamma : X \to A$ is **upper hemi-continuous and compact valued** (u.h.c.) at $x \in X$ if for every sequence $x_n \xrightarrow{n} x$ and every sequence $(a_n)_{n \geq N}$ such that $a_n \in \Gamma(x_n)$ for all n, the sequence $(a_n)_{t \in \mathbb{N}}$ is bounded and if $(a_n)_{n \in N}$ converges, then its limit point a is in $\Gamma(x)$.*

Definition 10 (continuity). *A correspondence $\Gamma : X \to A$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c. at x. A correspondence is continuous if it is continuous at each point in its domain.*

[29] A correspondence $\Gamma : X \to A$ is a mathematical object that takes a vector $x \in X$ and delivers a subset $\Gamma(x) \subseteq A$.

[30] Here, compactness means that for each $x \in X$, $\Gamma(x)$ is closed and bounded.

[31] This follows from the extreme value theorem.

[32] If $\Gamma(x_n)$ is nonempty for all n, we can always take $N = 1$.

The following lemma is a well known result concerning convergence of sequences and will be useful in the proof of the following theorem.

Lemma 3. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. The function v is continuous at x if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \xrightarrow{n} x$ there is a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ such that $v(x_{\varphi(n)}) \xrightarrow{n} v(x)$.*

Proof. ($\to$) Let v be continuous and $x_n \xrightarrow{n} x$. Then evidently $v(x_n) \xrightarrow{n} v(x)$ so for all subsequences $(x_{\varphi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we should have that $v(x_{\varphi(n)}) \xrightarrow{n} v(x)$.

($\leftarrow$) For the reverse, assume that v is not continuous at x. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $v(x_n) \not\xrightarrow{n} v(x)$. From this, we will construct a sequence $z_t \to x$ that has no subsequence $z_{\varphi(n)}$ such that $v(z_{\varphi(n)})$ converges to $v(x)$.

As $v(x_n) \not\to v(x)$, there exists a $\varepsilon > 0$ such that for all T, there is an $n \geq T$ such that $|v(x) - v(x_n)| > \varepsilon$. This generates a sequence of numbers $n_1, n_2, n_3, \ldots$ such that for all k, $|v(x) - v(x_{n_k})| > \varepsilon$. Without loss of generality, we can assume that this sequence of numbers is increasing. Let $z_k = x_{n_k}$. We see that $(z_n)_{n\in N}$ is a sequence such that $z_n \xrightarrow{n} x$. Additionally, $(v(z_n))_{n\in\mathbb{N}}$ has no subsequence $(v(z_{\varphi(n)}))_{n\in\mathbb{N}}$ that converges to $v(x)$, as was to be shown. □

Theorem 4 (Theorem of the maximum)**.** *Let $X \subseteq \mathbb{R}^l$, $A \subseteq \mathbb{R}^m$. Let $f : X \times A \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to X$ be a continuous correspondence. Then the function $v : X \to \mathbb{R}$,*

$$v(x) = \max_{a\in\Gamma(x)} f(x,a).$$

is continuous, and the correspondence $G : X \to X$

$$G(x) = \{a \in \Gamma(x) : f(x,a) = v(x)\},$$

is non-empty and u.h.c.

Proof. Let us first show that v is continuous. By the above lemma, it suffices to show that any sequence $x_n \to x$ has a subsequence $(x_{\varphi(n)})_{n\in N}$ such that $v(x_{\varphi(n)}) \xrightarrow{n} v(x)$. Take any $x \in X$ and consider any sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{n} x$. We need to construct a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ such that $v(x_{\varphi(n)}) \xrightarrow{n} v(x)$.

As $x_n \xrightarrow{n} x$ we have for all n an element $a_n \in G(x_n)$ and $v(x_n) = f(x_n, a_n)$. As Γ is u.h.c. we have that there exists a subsequence $(a_{\varphi(n)})_{n\in N}$ converging to $a \in \Gamma(x)$. Also, as f is continuous, $\lim_n f(x_{\varphi(n)}, a_{\varphi(n)}) = \lim_n v(x_{\varphi(n)}) = f(x,a)$. Let us finish the proof by showing that $f(x,a) = v(x)$. If not, then there is an element $a' \in G(x)$ such that $v(x) = f(x,a') > f(x,a)$.

We have that $a' \in \Gamma(x)$, so by l.h.c. we have that there is a sequence $a'_{\varphi(n)} \overset{n}{\to} a'$ such that $a'_{\varphi(n)} \in \Gamma(x_{\varphi(n)})$. By optimality of $a_{\varphi(n)}$ we have that for all $n \in \mathbb{N}$:

$$f(x_\varphi(n), a_\varphi(n)) \geq f(x_\varphi(n), a'_{\varphi(n)}).$$

Taking limits on both sides and using continuity of f gives $f(x,a) \geq f(x,a')$, which gives the contradiction.

Next, let us show that $G(x)$ is u.h.c. Fix x and let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x and let $a_n \in G(x_n)$. As $a_n \in \Gamma(x_n)$ for all n, we know that $(a_n)_{n\in\mathbb{N}}$ is bounded. Next if $a_n \overset{n}{\to} a$ then by continuity of v and f.

$$v(x) = \lim_n v(x_n) = \lim_n f(x_n, a_n) = f(x,a),$$

by continuity of f and v. As such, $a \in G(x)$ which shows that G is u.h.c. □

In general the optimal value correspondence is not l.h.c. Consider the example where $X = \mathbb{R}$, $f(x,a) = xa^2$ and $\gamma(x) = [-1,1]$ for all x. Then

$$G(x) = \begin{cases} \{-1,1\} \text{ if } x > 0 \\ [-1,1] \text{ if } x = 0, \\ 0 \text{ if } x < 0. \end{cases}$$

Take a sequence $x_t \to 0$ where $x_t < 0$ for all t. Then $0.5 \in G(0)$ but $G(x_t) = \{-1,1\}$ for all x_t so there is not sequence in $G(x_t)$ that converges to 0.5 which means that the correspondence G is not l.h.c. at $x = 0$.

Theorem 5. *Let $X \subseteq \mathbb{R}^l$ and $A \subseteq \mathbb{R}^k$. Let $\Gamma : X \to A$ be convex valued, compact valued and continuous. Assume that $f : X \times A \to \mathbb{R}$ is continuous and that $f(x,a)$ is strictly (quasi)-concave in its second argument then if*

$$v(x) = \max_{a\in\Gamma(x)} f(x,a),$$

we have that

$$g(x) = \{y \in \Gamma(x) : f(x,y) = v(x).\}$$

is single valued, and the function $g(x)$ is continuous.

Proof. From the theorem of the maximum, we know that the optimal solution correspondence G is bounded, compact valued and u.h.c. Let us first show that G is single valued. Assume, towards a contradiction, that $a_1, a_2 \in G(x)$, i.e. $f(x,a_1) = f(x,a_2) = v(x)$. Then $\lambda a_1 + (1-\lambda)a_2 \in \Gamma(x)$ for $\lambda \in (0,1)$ so:

$$f(x, \lambda a_1 + (1-\lambda)a_2) > \min\{f(x,a_1), f(x,a_2)\} = v(x).$$

but this contradicts the optimality of a_1 and a_2. This shows that g is a single valued function. For continuity, let $x_n \to x$. It suffices to show that $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ such that $g(x_{\varphi(n)}) \overset{n}{\to} g$. Obviously, $g(x_n) \in G(x_n)$. Then by u.h.c. of G, $(g(x_n))_{n\in\mathbb{N}}$ is bounded. So it has a convergent subsequence, say $g(x_{\varphi(n)}) \overset{n}{\to} g$. Also $g(x_{\varphi(n)}) \in G(x_{\varphi(n)})$ so again by u.h.c., $g \in G(x)$. By single valuedness of G, we have $g = g(x)$. □

The second part of the theorem is actually true in general. If g is a function and u.h.c. then it is continuous. Also if g is a function and l.h.c. then it is also continuous.

Theorem 6. *Let $X \subseteq \mathbb{R}^l$, $A \subseteq \mathbb{R}^k$. Let $\Gamma : X \to A$ be convex valued, compact valued and continuous. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $X \times A$ and assume that for all n, $f_n(x,a)$ is strictly concave in its second argument. Assume that f has the same properties and that $\|f_n - f\|_\phi \overset{n}{\to} 0$. Let,*

$$g_n = \arg\max_{a\in\Gamma(x)} f_n(x,a),$$

and

$$g = \arg\max_{a\in\Gamma(x)} f(x,a).$$

then for all $x \in X$, $g_n(x) \overset{n}{\to} g(x)$. If X is compact then $\|g_n - g\|_\phi \to 0$.

Proof. We have that,

$$\begin{aligned}
0 &\le f(x,g(x)) - f(x,g_n(x)),\\
&\le (f(x,g(x)) - f(x,g_n(x))) + \underbrace{(f_n(x,g_n(x)) - f_n(x,g(x)))}_{\ge 0},\\
&= (f(x,g(x)) - f_n(x,g(x)) + (f_n(x,g_n(x)) - f(x,g_n(x))),\\
&\le \phi(x)\|f - f_n\|_\phi + \phi(x)\|f_n - f\|_\phi.
\end{aligned}$$

This shows that:

$$\sup_{x\in X}\left|\frac{f(x,g(x)) - f(x,g_n(x))}{\phi(x)}\right| \overset{n}{\to} 0.$$

First, to show that for all $x \in X$, $\|g_n(x) - g(x)\| \overset{n}{\to} 0$, assume, towards a contradiction that for some $x \in X$, $g_n(x) \overset{n}{\not\to} g(x)$. If so, there must be a subsequence $g_{\varphi(n)}(x)$ and a $\varepsilon > 0$ such that for all n,

$$\|g_{\varphi(n)}(x) - g(x)\| \ge \varepsilon.$$

Let,

$$A_\varepsilon = \{a \in \Gamma(x) : \|a - g(x)\| \ge \varepsilon\}.$$

We know that for all n, $g_{\varphi(n)}(x) \in A_\varepsilon$, so A_ε is non-empty. Also the element $g(x) \notin A_\varepsilon$. The set A_ε is also a compact subset of A.

Let

$$\delta = \min_{a\in A_\varepsilon} |f(x,g(x)) - f(x,a)|.$$

This problem is well defined as A_ε is a compact set and the objective function is continuous. Also $f(x,g(x)) \ge f(x,a)$ for all $a \in A_\varepsilon \subseteq \Gamma(x)$. In fact, the absolute value $|f(x,g(x)) - f(x,a)|$ is equal to 0 only if a solves the maximization problem which means that in this case $g(x) = a$. However $g(x) \notin A_\varepsilon$ which means that the solution to this minimization problem must give $\delta > 0$. As $g_{\varphi(n)}(x) \in A_\varepsilon$ it follows that for all n,

$$|f(x,g(x)) - f(x,g_{\varphi(n)}(x))| > \delta.$$

However $|f(x,g(x)) - f(x,g_{\varphi(n)}(x))| \to 0$ as $\sup_{x\in X}\left|\frac{f(x,g(x))-f(x,g_n(x))}{\phi(x)}\right| \overset{n}{\to} 0$, a contradiction.

For the second part, assume that X is compact. Let us show that $\|g_n - g\|_\phi \overset{n}{\to} 0$. If not then there exists a subsequence $(g_{\varphi(n)})_{n\in\mathbb{N}}$ such that for all n,

$$\|g_{\varphi(n)} - g\|_\phi \geq \varepsilon.$$

In particular, for all n there exists an $x_n \in X$ such that

$$\frac{\|g_{\varphi(n)}(x_n) - g(x_n)\|}{\phi(x_n)} \geq \frac{\varepsilon}{2}.$$

Let,

$$A_\varepsilon = \left\{(x,a) \in X \times A : a \in \Gamma(x), \frac{\|a - g(x)\|}{\phi(x)} \geq \frac{\varepsilon}{2}\right\}.$$

One sees that A_ε is a compact subset of $X \times A$. For all n, there is an $x_n \in X$ and an $g_{\varphi(n)}(x_n) \in \Gamma(x_n)$, such that $(x_n, g_{\varphi(n)}(x_n)) \in A_\varepsilon$, so it is non-empty. Finally, for all $x \in X$, $(x,g(x)) \notin A_\varepsilon$.

Let,

$$\delta = \min_{(x,a)\in A_\varepsilon} \frac{|f(x,g(x)) - f(x,a)|}{\phi(x)}.$$

Observe that the objective function is zero only if $f(x,g(x)) = f(x,a)$ which only happens if $a = g(x)$. However, we know that $(x,g(x)) \notin A_\varepsilon$. This implies that $\delta > 0$. As for all n, $(x_n, g_{\varphi(n)}(x_n)) \in A_\varepsilon$, we therefore have that for all n:

$$\frac{|f(x_n,g(x_n)) - f(x_n,g_{\varphi(n)}(x_n))|}{\phi(x_n)} \geq \delta > 0.$$

This contradicts the fact that,

$$\sup_{x\in X}\left|\frac{f(x,g(x)) - f(x,g_n(x))}{\phi(x)}\right| \overset{n}{\to} 0.$$

□

Dynamic programming under certainty

In this chapter we will investigate the infinite horizon optimization problem that we presented in chapter 1.

We denote by $X \subseteq \mathbb{R}^l$ the state space. An element $x \in X$ captures the state of the world at a particular point in time. We denote by $A \subseteq \mathbb{R}^k$ the set of controls variables and we denote by $\Gamma : X \to A$ the correspondence that determines for all states x the possible values of the control variable. The next period feasible states are determined by a function $r : X \times A \to X$ such that $y = r(x,a) \in X$ is the next period's state if the current state is x and the chosen control variable has the value a. We also denote by $F : X \times A \to \mathbb{R}$ the instantaneous payoff function that depends on the values of the current state and control. Finally, let $\beta \in (0,1)$ be a discount factor. In this section, we will be interested in finding solutions to the following infinite horizon optimization problem,

$$\begin{aligned} v(x_0) = \max_{a_0,a_1,a_2,\ldots} & \sum_{t=0}^{\infty} \beta^t F(x_t, a_t), \\ \text{s.t. } & x_{t+1} = r(x_t, a_t), \\ & a_t \in \Gamma(x_t), \\ & x_0 \text{ given.} \end{aligned}$$

We will do this by relating it to the fixed point of the so called Bellman operator, $T : B_\phi(X) \to B_\phi(X)$ defined by,

$$(Tv)(x) = \max_{a \in \Gamma(x)} \{F(x,a) + \beta v(r(x,a))\}.$$

In particular, we will show that under certain conditions the first problem has a solution, and its solution is equivalent to the fixed point of the Bellman operator Tv.

Definition 11 (Regularity condition)**.** *The problem (X, A, Γ, F, β) is **regular** if the one period return function $F : X \times A \to \mathbb{R}$ and the transition function $r : X \times A \to \mathbb{R}$ are continuous, if the transition correspondence $\Gamma : X \to A$ is non-empty, continuous (u.h.c. and l.h.c.) and, additionally, there is a continuous function $\phi : X \to \mathbb{R}_{++}$ such that,*

1. *There exists an $M \geq 0$ such that for all $x \in X$:*

$$\max_{a\in\Gamma(x)} |F(x,a)| \leq M\phi(x).$$

2. *There exists a $\theta \in (0,1)$ such that for all $x \in X$:*

$$\beta \max_{a\in\Gamma(x)} \phi(r(x,a)) \leq \theta\phi(x).$$

Observe that if the function F is bounded, then the regularity conditions are satisfied by choosing $\phi(x) = 1$ for all x.

Theorem 7. *If the problem (X, A, Γ, F, β) is regular then the Bellman operator is a contraction mapping from $B_\phi(X)$ to $B_\phi(X)$.*

Proof. If $v \in B_\phi(X)$, then v is continuous. Also F and r are continuous and Γ is continuous and compact valued. As such, the optimization problem,

$$(Tv)(x) = \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\},$$

is well defined for all $x \in X$. By the theorem of the maximum, the maximum value function is continuous in x. As such, $(Tv)(x)$ is a continuous function of x. in order to show that $T : B_\phi(X) \to B_\phi(X)$ it suffices to show that $\|Tv\|_\phi$ is bounded whenever $\|v\|_\phi$ is bounded. Now, assume that $\|v\|_\phi < N$ then,

$$\begin{aligned}
|(Tv)(x)| &= \left|\max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\}\right|, \\
&\leq \max_{a\in\Gamma(x)} |F(x,a)| + \beta \max_{a\in\Gamma(x)} |v(r(x,a))|, \\
&\leq M\phi(x) + \beta N \max_{a\in\Gamma} \phi(r(x,a)), \\
&\leq M\phi(x) + N\theta\phi(x) = (M + N\theta)\phi(x).
\end{aligned}$$

This shows that $\|Tv\|_\phi \leq M + N\theta$, so $\|Tv\|_\phi$ is bounded. Conclude that $Tv \in B_\phi(X)$.

In order to show that T is a contraction mapping, we use Blackwell's theorem. For monotonicity, assume that $v \geq w$. Then,

$$\begin{aligned}
(Tv)(x) &= \max_{a\in\Gamma(x)} F(x,a) + \beta v(r(x,a)), \\
&\geq \max_{a\in\Gamma(x)} F(x,a) + \beta w(r(x,a)) = (Tw)(x).
\end{aligned}$$

For additivity,

$$\begin{aligned}
(T(v + a\phi))(x) &= \max_{a\in\Gamma(x)} \{F(x,a) + \beta\,(v + a\phi)\,(r(x,a))\}, \\
&\leq \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\} + a \max_{a\in\Gamma(x)} \beta\phi(r(x,a)), \\
&\leq (Tv)(x) + a\theta\phi(x).
\end{aligned}$$

□

Above theorem shows that the Bellman operator has a fixed point, say v, which is continuous and bounded in the $\|.\|_\phi$ norm. Associated with the Bellman operator, we can find a policy correspondence, G where

$$G(x) = \{a \in \Gamma(x) : F(x,a) + \beta v(r(x,a)) = v(x)\}.$$

Consider our infinite horizon optimization problem.

$$\begin{aligned} &\sup_{(a_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, a_t), \\ &\text{s.t. } a_t \in \Gamma(x_t), \\ &\quad x_{t+1} = r(x_t, a_t), \\ &\quad x_0 \in X \text{ given.} \end{aligned}$$

When is this problem well defined? When can we replace the sup with a max operator? When does the solution coincide with the fixed point of the Bellman operator? These are three questions that we are going to answer now.

Definition 12 (feasible path)**.** *A **feasible path** is a sequence $((x_0, a_0), (x_1, a_1), \ldots)$ such that for all $t \in \mathbb{N}$,*

1. $a_{t-1} \in \Gamma(x_{t-1})$,
2. $x_t = r(x_{t-1}, a_{t-1})$.

Let $\Pi(x_0)$ be the set of all feasible paths that start at the state $x_0 \in X$. Then for such paths $p = ((x_0, a_0), (x_1, a_1), \ldots) \in \Pi(x_0)$ we define,

$$w_p = \sum_{t=0}^{\infty} \beta^t F(x_t, a_t) \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, a_t),$$

whenever this limit is well defined.

Lemma 4. *Assume that (X, Γ, F, β) is regular and let $x_0 \in X$. Then for all paths $p \in \Pi(x_0)$, w_p is well defined and the set $\{w_p : p \in \Pi(x_0)\}$ is bounded from above.*

Proof. Fix a feasible path $p = \{(x_0, a_0), (x_1, a_1), \ldots, \} \in \Pi(x_0)$ and define $u_n = \sum_{t=0}^{n} \beta^t F(x_t, a_t)$. In order to show that w_p is well defined, we need to show that the sequence $u_1, u_2, u_3, \ldots$ converges. We do this by showing that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.[33] Fix $\varepsilon > 0$. We have to find an $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$.

$$|u_n - u_m| < \varepsilon.$$

[33] Remember that $\mathbb{R}$ is a Banach space, so every Cauchy sequence of real numbers converges.

Assume w.l.o.g. that $n > m$ then,

$$
\begin{aligned}
|u_n - u_m| &= \left| \sum_{t=m}^{n} \beta^t F(x_t, a_t) \right|, \\
&\leq \sum_{t=m}^{n} \beta^t |F(x_t, a_t)|, \\
&\leq \sum_{t=m}^{n} M\beta^t \phi(x_t).
\end{aligned}
$$

Now, $\phi(x_t) \leq \max_{a\in\Gamma(x_{t-1})} \phi(r(x_{t-1}, a)) \leq \frac{\theta}{\beta}\phi(x_{t-1})$. Iterating this further, we see that, $\phi(x_t) \leq \left(\frac{\theta}{\beta}\right)^t \phi(x_0)$. As such,

$$
\begin{aligned}
|u_n - u_m| &\leq \sum_{t=m}^{n} M\theta^t \phi(x_0), \\
&\leq M\theta^m \phi(x_0) \sum_{t=0}^{n-m} \theta^t, \\
&\leq M\theta^m \frac{\phi(x_0)}{1-\theta} \overset{n}{\to} 0
\end{aligned}
$$

This shows that $(u_n)_{n\in\mathbb{N}}$ is Cauchy. So $u_n \overset{n}{\to} u \equiv w_p$ is well defined.

Now let us show that there is an A (which may depend on x_0) such that for all paths $p = \{(x_0, a_0), (x_1, a_1), \ldots\} \in \Pi(x_0)$, $w_p \leq A$.[34] As above, we have that,

[34] Remember: every set of real numbers which is bounded from above has a supremum, so this shows that $\sup\{w_p : p \in \Pi(x_0)\}$ is well defined.

$$
\beta^t |F(x_t, a_t)| \leq M\beta^t \phi(x_t) \leq M\theta^t \phi(x_0).
$$

This gives,

$$
\begin{aligned}
u_n &= \sum_{t=0}^{T} \beta^t F(x_t, a_t), \\
&\leq \sum_{t=0}^{T} \beta^t |F(x_t, a_t)|, \\
&\leq \sum_{t=0}^{T} \theta^t M\phi(x_0), \\
&\leq M\phi(x_0) \frac{1}{1-\theta}.
\end{aligned}
$$

Setting $A = \frac{1}{1-\theta} M\phi(x_0)$ demonstrates the proof. □

This lemma shows that $V(x_0) = \sup\{w_p : p \in \Pi(x_0)\}$ is well defined.

The next step is to show that the fixed point of the Bellman operator v satisfies that for all $x_0 \in X$, (i) $v(x_0) \geq V(x_0)$ and (ii) for all $x_0 \in X$, there is a path $p \in \Pi(x_0)$ such that $v(x_0) = w_p$.

Lemma 5. *Let (X, A, Γ, F, β) be a regular problem and let v be the fixed point of the Bellman operator. Then for all paths $p \in \Pi(x_0)$ $v(x_0) \geq w_p$.*[35]

[35] In other words, $v(x_0)$ is an upper bound for $\{w_p : p \in \Pi(x_0)\}$.

Proof. Let v be the fixed point of the Bellman operator and let $p = ((x_0, a_0), (x_1, a_1), \dots) \in \Pi(x_0)$ be a path. We will show that $v(x_0) \geq w_p$.

Now, by definition $v(x) = \max_{a \in \Gamma(x)} F(x, a) + \beta v(r(x, a))$, so,

$$\begin{aligned} v(x_0) &\geq F(x_0, a_0) + \beta v(x_1), \\ &\geq F(x_0, a_0) + \beta F(x_1, a_1) + \beta^2 v(x_2), \\ &\dots \\ &\geq \sum_{t=0}^{T} \beta^t F(x_t, a_t) + \beta^T v(x_T). \end{aligned}$$

Taking the limit for $T \to \infty$, gives[36],

$$v(x_0) \geq \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, a_t) + \lim_{T \to \infty} \beta^T v(x_T) = w_p + \lim_{T \to \infty} \beta^T v(x_T).$$

[36] Observe that the limit $\lim_T \sum_{t=0}^{T} \beta^t F(x_t, a_t)$ is well defined by the previous lemma.

As such, we only need to show that $\lim_{T \to \infty} \beta^T v(x_T) = 0$.[37] Now,

[37] In other words, for all $\varepsilon > 0$ there exists an N_ε such that for all $n \geq N_\varepsilon$, $|\beta^t v(x_t)| < \varepsilon$.

$$|v(x_T)| = \frac{|v(x_T)|}{\phi(x_T)} \frac{\phi(x_T)}{\phi(x_{T-1})} \cdots \frac{\phi(x_1)}{\phi(x_0)} \phi(x_0).$$

Also:

$$\frac{|v(x_T)|}{\phi(x_T)} \leq \|v\|_\phi.$$

And, for all t,

$$\begin{aligned} &\phi(x_t) \leq \max_{a \in \Gamma(x_{t-1})} \phi(r(a, x_{t-1}) \leq \frac{\theta}{\beta} \phi(x_{t-1}), \\ \to &\frac{\phi(x_t)}{\phi(x_{t-1})} \leq \frac{\theta}{\beta}. \end{aligned}$$

from this

$$\begin{aligned} &|v(x_T)| \leq \left(\frac{\theta}{\beta}\right)^{T-1} \|v\|_\phi \phi(x_0), \\ \to &\beta^T |v(x_t)| \leq \theta^{T-1} \beta \|v\|_\phi \phi(x_0) \xrightarrow{T} 0. \end{aligned}$$

This shows that the value function is an upper bound to any feasible solution of the infinite horizon optimization problem, so $v(x_0) \geq V(x_0)$. □

Given the fixed point of the Bellman operator, define recursively a path $g = ((x_0, a_0), (x_1, a_1), \dots) \in \Pi(x_0)$ by,[38]

[38] The right hand side is a maximization problem of a continuous function $(F(x_t, a) + \beta v(r(x_t, a))$ over a compact set $\Gamma(x_t)$ so the solution a_t and therefore the path g is well defined.

$$\begin{aligned} a_t &\in \arg\max_{a \in \Gamma(x_t)} \left\{ F(x_t, a) + \beta v(r(x_t, a)) \right\}, \\ x_{t+1} &= r(x_t, a_t). \end{aligned}$$

Lemma 6. *Let* (X, A, Γ, F, β) *be a regular problem and let* v *be the fixed point of the Bellman operator. Then for all* $x_0 \in X$, $v(x_0) = w_g$.

Proof. Let $g = ((x_0, a_0), (x_1, a_1), \ldots)$ be the path as defined above. Then, we have,

$$\begin{aligned} v(x_0) &= F(x_0, a_0) + \beta v(x_1), \\ &= F(x_0, a_0) + \beta F(x_1, a_1) + \beta v(x_2), \\ &\ldots, \\ &= \sum_{t=1}^{T} \beta^t F(x_t, a_t) + \beta^T v(x_T). \end{aligned}$$

Taking limits gives, $v(x_0) = w_g(x_0) + \lim_T \beta^T v(x_T)$ and similar to the proof of the previous lemma, we know that

$$\lim_T \beta^T v(x_T) = 0.$$

as was to be shown. □

Let us consider the following example,

$$\begin{aligned} \max_{c_0, c_1, \ldots} \sum_{t=0}^{\infty} \beta^t \ln(c_t + 1) &\text{ s.t. } k_{t+1} = k_t^\alpha - c_t, \\ \text{s.t. } 0 &\leq c_t \leq k_t^\alpha, \\ k_0 &\text{ given.} \end{aligned}$$

Here, capital k is the state variable and consumption c is the control variable, $\ln(c+1)$ is the instantaneous utility function, $\beta \in (0, 1)$ is the discount factor and δ is the depreciation rate. In terms of the model outlined above, we have that $X = \mathbb{R}$, $A = \mathbb{R}$, $F = \ln(c_t + 1)$, $\Gamma(k) = \{c \in \mathbb{R}_+ : 0 \leq c \leq k^\alpha\}$ and $r(k, c) = k^\alpha - c$. We assume that $\alpha \leq 1$.

The Bellman operator is given by,

$$(Tv)(k) = \max_{0 \leq c \leq k^\alpha} \ln(c+1) + \beta v(k^\alpha - c).$$

In order for the fixed point of the Bellman operator to solve the problem, we need to find a function $\phi(k) > 0$ a number M and a $\theta < 1$ such that,

- $\max_{0 \leq c \leq k^\alpha} \ln(c+1) \leq M\phi(k)$.
- $\beta \max_{0 \leq c \leq k^\alpha} \phi(k^\alpha - c) \leq \theta\phi(k)$.

Let's try the function $\phi(k) = \ln(k+1) + 1 > 0$ which is strictly positive and continuous. Consider first the case where $\alpha \leq 1$. Then for the first condition,

$$\max_{0\leq c\leq k^\alpha} |\ln(c+1)| = \ln(k^\alpha+1),$$
$$\leq \ln(k+1) \leq \ln(k+1)+1 = \phi(k).$$

The first inequality uses $k^\alpha \leq k$ and the fact that ln(.) is increasing. Setting $M = 1$ shows the condition. For the second condition,

$$\beta\left(\max_{0\leq c\leq k^\alpha} \ln((k^\alpha - c)+1)+1\right) = \beta\ln(k^\alpha+1)+\beta,$$
$$\leq \beta(\ln(k+1)+1) = \beta\phi(k),$$

Setting $\theta = \beta < 1$ shows the condition.

Let's now consider the case $\alpha > 1$. For the first condition,

$$\max_{0\leq c\leq k^\alpha} |\ln(c+1)| = \ln(k^\alpha+1),$$
$$\leq \ln((k+1)^\alpha) = \alpha\ln(k+1) \leq \alpha(\ln(k+1)+1).$$

The inequality follows from the fact that $k^\alpha + 1 \leq (k+1)^\alpha$ if $\alpha > 1$.[39] Setting $M = \alpha$ shows the condition. For the second condition,

[39] In order to see this, notice that $k^\alpha+1$ and $(k+1)^\alpha$ are equal when $k = 0$. However, the derivative of $(k+1)^\alpha$ is always bigger than that of $k^\alpha + 1$, so the functions start at the same point for $k = 0$ but the slope is always bigger for the first than for the second.

$$\beta\left(\max_{0\leq c\leq k^\alpha} (\ln((k^\alpha - c)+1)+1)\right) = \beta\ln(k^\alpha+1)+\beta,$$
$$\leq \beta\alpha\ln(k+1)+\beta,$$
$$\leq \beta\alpha(\ln(k+1)+1) = \beta\alpha\phi(k).$$

Setting $\theta = \beta\alpha$ shows the second condition. Observe that this requires that $\alpha < 1/\beta$ so α can be greater than one but not too high.

Properties of the Bellman fixed point

LET US NOW have a look at some of the properties of the fixed point of the Bellman operator. In this part, we will take the simplifying assumption that $A = X$ and $r(x, a) = a$. In other words, the optimization problem can be rewritten as,

$$v(x) = \max_{a\in\Gamma(x)} F(x,a) + \beta v(a).$$

Theorem 8. *Let (X, Γ, F, β) satisfy the assumptions of Definition 11. Let $F(., a)$ be strictly increasing in each of its first arguments and assume that Γ is monotone in the sense that for $x \leq x'$,*

$$\Gamma(x) \subseteq \Gamma(x').$$

Then, the fixed point v of the Bellman operator is strictly increasing.

Proof. Let $B'_\phi(X) \subseteq B_\phi(X)$ be the set of bounded (in the $\|.\|_\phi$ norm), continuous, weakly-increasing functions on X and let $B''_\phi(X) \subset B'_\phi(X)$ be the subset of strictly increasing functions. Since $B'_\phi(X)$ is a closed subset of $B_\phi(X)$ it suffices to show that $T[B'_\phi(X)] \subseteq B''_\phi(X)$.[40]

[40] In other words, the Bellman operator maps weakly increasing functions into the set of strictly increasing functions.

If $x' > x$ then by monotonicity of Γ: $\Gamma(x) \subseteq \Gamma(x')$. Let a' solve $\max_{a\in\Gamma(x)}\{F(x,a) + \beta v(a)\}$. Then,

$$\begin{aligned}(Tv)(x) = F(x,a') + \beta v(a')) &< F(x',a') + \beta v(a'),\\ &\leq \max_{a\in\Gamma(x')} F(x',a) + \beta v(a) = (Tv)(x').\end{aligned}$$

□

Theorem 9. *Let (X, Γ, F, r, β) satisfy the assumptions of Definition 11. Let F be strictly concave, i.e. for all $\theta \in (0,1)$,*

$$F(\theta(x,a) + (1-\theta)(x',a')) \geq \theta F(x,a) + (1-\theta)F(x',a'),$$

with a strict inequality if $(x,a) \neq (x',a')$ and assume that Γ is convex in the sense that for all $\theta \in [0,1]$ and $x, x' \in X$,

$$a \in \Gamma(x), a' \in \Gamma(x') \text{ implies } \theta a + (1-\theta)a' \in \Gamma(\theta x + (1-\theta)x').$$

Then v is strictly concave and $g(x) = \arg\max_{a\in\Gamma(x)} F(x,a) + \beta v(a)$ is a continuous, single-valued function.

Proof. Let $B'_\phi(X) \subseteq B_\phi(X)$ be the set of bounded continuous, weakly concave functions and let $B''_\phi(X) \subseteq B'_\phi(X)$ be the subset of strictly concave functions. It suffices to show that $T[B'_\phi(X)] \subseteq B''_\phi(X)$. Let v be concave and let $x_0 \neq x_1$, $\theta \in (0,1)$ and set $x_\theta = \theta x_0 + (1-\theta)x_1$. Also let a_0 solve $\max_{a\in\Gamma(x_0)} F(x_0,a) + \beta v(a)$ and a_1 solve $\max_{a\in\Gamma(x_1)} F(x_1,a) + \beta v(a)$. Let $a_\theta = \theta a_0 + (1-\theta)a_1$. Then,

$$\begin{aligned}(Tv)(x_\theta) &\geq F(x_\theta, a_\theta) + \beta v(a_\theta),\\ &> \theta F(x_0,a_0) + (1-\theta)F(x_1,a_1) + \beta\theta v(a_0) + \beta(1-\theta)v(a_1),\\ &= \theta(Tv)(x_0) + (1-\theta)(Tv)(x_1).\end{aligned}$$

This shows that the Belman fixed point function is strictly concave. Given strict concavity,

$$\max_{a\in\Gamma(x)} F(x,a) + \beta v(a),$$

maximizes a strictly concave function. As such, the optimal value is unique so $g(x)$ is a function. As g is also u.h.c. (from Berge's maximum theorem), so the function g is continuous. □

Theorem 10. *Let (X, Γ, F, β) satisfy the assumptions of Definition 11 and let v be the fixed point of the Bellman operator. Let $F(x,a)$ be strictly* *concave in a, let $B'_\phi(X)$ be the set of bounded continuous, concave functions and let $v_0 \in B'_\phi(X)$. Let $(v_n, g_n)_{n\in\mathbb{N}}$ be defined as,*

$$\begin{aligned} v_{n+1} &= Tv_n, \\ g_n(x) &= \arg\max_{a\in\Gamma(x)} F(x,a) + \beta v_n(a). \end{aligned}$$

Then $g_n \to g$ pointwise. If X is compact, then $\|g_n - g\|_\phi \to 0$.

Proof. Let $B''_\phi(X)$ be the set of strictly concave bounded continuous functions. We know that for all n, $v_n \in B''_\phi(X)$. For $a \in \Gamma(x)$, let $f_n(x,a) = F(x,a) + \beta v_n(a)$. We have that every function $f_n(x,y)$ is strictly concave. Also let $f(x,a) = F(x,a) + \beta v(a)$. Then,

$$\begin{aligned} |f_n(x,a) - f(x,a)| &= \beta |v_n(a) - v(a)|, \\ &= \phi(a)\beta \frac{|v_n(a) - v(a)|}{\phi(a)}, \\ &\le \phi(x)\theta \|v_n - v\|_\phi \xrightarrow{n} 0 \end{aligned}$$

This shows that $\|f_n(x,a) - f(x,a)\|_\phi \xrightarrow{n} 0$. As such $g_n(x) \xrightarrow{n} g(x)$ pointwise. If X is compact, we get that $\|g_n - g\|_\phi \xrightarrow{n} 0$. □

The following part provides assumptions for which the value function can be assumed to be differentiable. It uses the Benveniste Scheinkman theorem.

Theorem 11 (Benveniste and Scheinkman). *Let $X \subseteq \mathbb{R}^l$ be a convex set, let $V : X \to \mathbb{R}$ be concave, let x_0 be in the interior of X and let D be a neighbourhood of x_0. If there is a concave, differentiable function $W : X \to \mathbb{R}$ with $W(x_0) = V(x_0)$ and $W(x) \le V(x)$ for all $x \in D$ then V is differentiable at x_0 and,*

$$\nabla_x V(x)|_{x=x_0} = \nabla_x W(x)|_{x=x_0}.$$

Proof. Any subgradient p of $V(x_0)$ must satisfy for all $x \in D$,

$$W(x) - W(x_0) \le V(x) - V(x_0) \le p(x - x_0).$$

This shows that p is also a subgradient of W. Since W is differentiable, the vector p must be unique and $p = \nabla_x W(x)|_{x=x_0}$. This means that V also has a unique subgradient and,

$$\nabla_x W(x)|_{x=x_0} = p_i = \nabla_x V(x)|_{x=x_0}.$$

□

Theorem 12. *Let (X, Γ, F, β) satisfy assumptions of theorem 11 and assume that F is strictly concave and Γ is convex. Assume that F is C^1. Let v be the* *fixed point of the Bellman operator and let g be the unique optimal value function. If x_0 is in the interior of X and $g(x_0)$ is in the interior of $\Gamma(x_0)$, then v is C^1 at x_0 and*

$$\nabla_x v(x)|_{x=x_0} = \nabla_x F(x,a)|_{(x,a)=(x_0,g(x_0))}.$$

Proof. As F is strictly concave and Γ is convex, g is a function. Also, since $g(x_0)$ is in the interior of $\Gamma(x_0)$ and Γ is continuous, $g(x_0)$ is in the interior of $\Gamma(x)$ for all x in a neighborhood D of x_0. Define W on D by

$$W(x) = F(x, g(x_0)) + \beta v(g(x_0)).$$

Since F is concave and differentiable, this function is concave and differentiable in x. Also,

$$\begin{aligned} W(x) &= F(x, g(x_0)) + \beta v(g(x_0)), \\ &\leq \max_{a\in\Gamma(x)} F(x,a) + \beta v(a) = v(x). \end{aligned}$$

with equality at x_0. The inequality uses the fact that $g(x_0) \in \Gamma(x)$ for all $x \in D$. The result follows from Benveniste and Sheinkman theorem. □

Euler equations

There is a second more classical mode of attack on the dynamic optimization problem. This alternative approach is based on first order conditions. As before, consider the following infinite horizon optimization problem.

$$\max_{x_1,x_2,\dots} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ s.t. } x_{t+1} \in \Gamma(x_t).$$

Assume that this problem can be solved and has a unique optimal solution $(x_n^*)_{n\in\mathbb{N}}$. Now, consider the following problem,

$$\max_{x_{t+1}} F(x_t^*, x_{t+1}) + \beta F(x_{t+1}, x_{t+2}^*) \text{ s.t. } x_{t+1} \in \Gamma(x_t^*), x_{t+2}^* = \Gamma(x_{t+1}).$$

Given the optimality of x_t^* and x_{t+2}^*, we need that x_{t+1}^* is the optimal solution to this problem. If F is differentiable and if x_{t+1}^*, x_{t+2}^* are in the interior of $\Gamma(x_t^*)$ and $\Gamma(x_{t+1}^*)$ (so the constraints are not binding), then we obtain the following first order condition:

$$\nabla_{x_{t+1}} F(x_t^*, x_{t+1}^*) + \beta \nabla_{x_{t+1}} F(x_{t+1}^*, x_{t+2}^*) = 0.$$

This equation is known as the Euler equation. If the solution is interior, then this condition is necessary for optimality. Usually the set of Euler equations is completed by adding a so called **transversality condition** namely,

$$\lim_{t\to\infty} \beta^t \nabla_{x_t} F(x_t^*, x_{t+1}) \cdot x_t^* \leq 0.$$

It can be shown that if the Euler equations are satisfied, F is concave, $\nabla_x F(x,y) > 0$, $x \geq 0$, and the transversality hold, then it must be that the solution $(x_0^*, x_1^*, \ldots, x_n^*, \ldots)$ is indeed optimal. To see this, let $(x_0, x_1, x_2, \ldots)$ be another feasible path. If F is concave, then:

$$\begin{aligned}\sum_{t=0}^{T} \beta^t \left(F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)\right) &\leq \sum_{t=0}^{T} \beta^t \nabla_{x_t} F(x_t^*, x_{t+1}^*) \cdot (x_t - x_t^*) + \sum_{t=0}^{T} \beta^t \nabla_{x_{t+1}} F(x_t^*, x_{t+1}^*) \cdot (x_{t+1} - x_{t+1}^*), \\ &= \nabla_{x_0} F(x_0^*, x_1^*)(x_0 - x_0^*) + \sum_{k=0}^{T-1} \beta^{k+1} \nabla x_{k+1} F(x_{k+1}, x_{k+2}) \cdot (x_{k+1} - x_{k+1}^*), \\ &+ \sum_{t=0}^{T} \beta^t \nabla_{x_{t+1}} F(x_t^*, x_{t+1}^*) \cdot (x_{t+1} - x_{t+1}^*),\end{aligned}$$

The first term on the right hand side is zero as $x_0^* = x_0$ is fixed. Then rearranging terms gives,

$$\begin{aligned}\sum_{t=0}^{T} \beta^t (F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)) &\leq \sum_{t=0}^{T-1} \beta^t \left[\beta \nabla x_{t+1} F(x_{t+1}^*, x_{t+2}^*) + \nabla x_{t+1} F(x_t^*, x_{t+1}^*)\right] \cdot (x_{t+1} - x_{t+1}^*), \\ &+ \beta^T \nabla x_{T+1} F(x_T^*, x_{T+1}^*) \cdot (x_{T+1} - x_{T+1}^*).\end{aligned}$$

The summations in the first line is equal to zero by the Euler equations. Then:

$$\begin{aligned}\sum_{t=0}^{T} \beta^t (F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*)) &\leq \beta^T \nabla_{x_{T+1}} F(x_T^*, x_{T+1}^*) \cdot (x_{T+1} - x_{T+1}^*), \\ &= \beta^T \left(-\beta \nabla_{x_{T+1}} F(x_{T+1}^*, x_{T+2}^*)\right) \cdot (x_{T+1} - x_{T+1}^*) \\ &\leq \beta^{T+1} \nabla_{x_{T+1}} F(x_{T+1}^*, x_{T+2}^*) \cdot x_{T+1}^*.\end{aligned}$$

where we used the Euler equation, the fact that $\nabla_x F(x,y) \geq 0$ and $x_t \geq 0$ for all T. Taking $T \to \infty$, the solution is optimal whenever:

$$\lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}) x_T^* \leq 0.$$

which is indeed the case if the transversality condition holds.

As an example, consider the Bellman equation of the optimal growth problem,

$$v(k) = \max_{k' \leq Ak^\alpha} \ln(Ak^\alpha - k') + \beta v(k').$$

The first order condition gives,

$$\frac{1}{k' - Ak^{\alpha}} + \beta v'(k).$$

Then from the envelope theorem we get,

$$v'(k) = \frac{A\alpha k^{\alpha-1}}{k' - Ak^{\alpha}}$$

Updating one period and substitution gives the Euler equation,

$$\frac{-1}{k_{t+1} - Ak_t^{\alpha}} + \beta \frac{\alpha A k_{t+1}^{\alpha-1}}{k_{t+2} - Ak_{t+1}^{\alpha}} = 0.$$

Which is a second order difference equation. The transversality condition requires that,

$$\lim_{t\to\infty} \beta^t \frac{A\alpha k_t^{\alpha}}{k_{t+1} - Ak_t^{\alpha}} \leq 0.$$

Numerical methods

IN THE PREVIOUS CHAPTER we say that the solution of the infinite horizon dynamic programming problem could be restated in terms of a solution of the Bellman equation. In many cases there are no closed form solutions to this Bellman equation[41] On the other hand, we have also seen that the unique solution of the Bellman equation coincides as the fixed point of a contraction mapping: the Bellman operator. Additionally, this fixed point can be approximated very precisely by iterating over this operator. In this sense, it is possible to approximate this fixed point via simulation methods. These methods are of course finite in nature and provide therefore only an approximation to the true fixed point.

[41] If there are solutions, they are mainly used for pedagogical purposes and are only available in a few special settings.

The simplest method is the value function iteration.

Value function iteration

THE CONTRACTION MAPPING theorem tells us that the solution of the Bellman equation can be found by iterating the Bellman operator T

$$(Tv)(x) = \max_{a \in \Gamma(x)} \{F(x, a) + \beta v(r(x, a))\}.$$

As computers only work with finite things, a first step is to approximates the state space X using a finite grid for the possible values of x,

$$X = \{x_1, x_2, \ldots, x_n\}.$$

Also the space of all possible actions A must be approximated using a finite grid:

$$A = \{a_1, \ldots, a_m\}.$$

The correspondence $\Gamma : X \to A$ is now replaced by a non-empty correspondence from the finite set X to the finite set A. Also, the instantaneous return function $F : X \times A \to \mathbb{R}$ is now a function from the finite set $X \times A$ to $\mathbb{R}$, so it is bounded by definition. Finally, the updating rule $r(x, a)$ must take values from $X \times A$ to the finite set X. It can be shown that when restricted to such finite setting, the Bellman operator T is still is a contraction mapping.[42] As such, finding, or approximating, the fixed point of T takes the following steps:

1. Decide on a (fine enough) grid for the state space X and control space A.[43]
2. Decide on some tolerance level $\varepsilon > 0$.[44]
3. Decide on an initial bounded function $v_0 : X \to \mathbb{R}$. Initiate the iteration round $t = 0$.
4. (a) Compute for all x in the finite grid X:

$$v_{t+1}(x) = \max_{a \in \Gamma(x)} \{F(x, a) + \beta v_t(r(x, a))\}.$$

So for given x one can compute for all $a \in \Gamma(x)$ the value $F(x, a) + \beta v_t(r(x, a)))$ and then take the maximum over all $a \in \Gamma(x)$. Given this value of a, we can save the policy correspondence:

$$G_{t+1}(x) = \arg\max_{a \in \Gamma(x)} \{F(x, a) + \beta v_t(r(x, a))\}.$$

(b) Repeat as long as $\|v_{t+1} - v_t\| = max_{x \in X} |v_{t+1}(x) - v_t(x)| > \varepsilon$, each time updating the counter $t \leftarrow t + 1$.

5. The final update gives a function v_t and policy correspondence G_t that should be a good approximation to the fixed point of the Bellman operator and the corresponding policy function.

In order to get a better grasp of the algorithm, let us work out a particular example. Consider a representative consumer model with CRRA utility function,[45]

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}.$$

The consumer maximizes her infinite horizon discounted utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t).$$

There is a stock of capital k_t in period t that she can use to produce an amount of capital in the next period using a production function $f(k_t) = k_t^\alpha$. There is also a depreciation rate of δ. This gives the following law of motion:

$$k_{t+1} = k_t^\alpha - c_t + (1 - \delta) k_t.$$

[42] Observe that F is bounded, so the conditions of Definition 11 are satisfied with $\phi(x) = 1$ which means that the Bellman operator $T : B(X) \to B(X)$ where $B(X)$ is the set of bounded functions on X has a unique fixed point by Blackwell's theorem. Also, the corresponding policy correspondence:

$$G(x) = \arg\max_{a \in \Gamma(x)} F(x, a) + v(r(x, a)),$$

is non-empty.

[43] Often the problem can be reformulated such that the control space and state space coincide, i.e. $A = X$.

[44] This should be sufficiently small.

[45] CRRA stands for constant relative risk aversion. The relative risk aversion of the utility function $u(.)$ is given by,

$$-\frac{u''(c)}{u'(c)} c.$$

As such, we obtain the following dynamic program:

$$\begin{aligned}
\max_{c_0,c_1,\dots} & \sum_{t=0}^{\infty} \beta^t u(c_t), \\
\text{s.t. } & k_{t+1} = k_t^{\alpha} - c_t + (1-\delta)k_t, \\
& c_t \in [0, k_t^{\alpha} - (1-\delta)k_t], \\
& k_0 \text{ given}
\end{aligned}$$

To make our lives a bit easier, we cancel the variable c_t from this problem:

$$\begin{aligned}
\max_{k_1,k_2,\dots} & \sum_{t=0}^{\infty} \beta^t u(k_t^{\alpha} + (1-\delta)k_t - k_{t+1}), \\
\text{s.t. } & k_{t+1} \in [0, k_t^{\alpha} + (1-\delta)k_t], \\
& k_0 \text{ given}
\end{aligned}$$

The Bellman operator for this problem is given by:

$$(Tv)(k) = \max_{k' \in [0, k^{\alpha} + (1-\delta)k]} \left\{ u(k^{\alpha} + (1-\delta)k - k') + \beta v(k') \right\}.$$

In particular, using the CRRA utilty function, we obtain:

$$(Tv)(k) = \max_{k' \in [0, k^{\alpha} + (1-\delta)k]} \left\{ \frac{(k^{\alpha} + (1-\delta)k - k')^{1-\sigma} - 1}{1-\sigma} + \beta v(k') \right\}.$$

Our aim is to write a program that computes the fixed point of T by sequentially computing $v_1 = Tv_0, v_2 = Tv_1, \dots$

1. First of all, we need to initialize some parameters. Let's pick the values:

   $$\sigma = 1.5, \qquad \delta = 0.1, \qquad \beta = 0.95, \qquad \alpha = 0.3.$$

   We also need to set the threshold for convergence which is a small number, say $\varepsilon = 10^{-3}$.

2. Next, we need to decide on a grid size, N, say $N = 1000$.

3. The grid size determines the number of values that we consider for our state variable, i.e. capital stock. As such, we intialize a vector $K = [k_1, \dots, k_N]$ of size 1000, say equally spaced between 0 and 5. The vector K represents our state space.

4. Next, we need to initialize the value function v and the updated value function Tv. These two things can easily be enoded using N-dimensional vectors $V = [v_1, \dots, v_N]$ where $V[i] \equiv v(k_i)$ gives the value of v at state k_i and $TV = [Tv_1, \dots, Tv_N]$ where $TV[i] \equiv Tv(k_i)$ gives the value of the function (Tv) at state k_i.

5. Finally, we need to encode the policy correspondence (or function) g. We will do this by representing g as an N-dimensional vector of integers $G = [g_1, \ldots, g_N]$. The idea is that the ith component of g is equal to j, i.e. $G[i] = j$ if the value of g at state k_i is given by k_j, i.e. $g(k_i) = k_j$. In other words, if $G[i] = j$ then at k_i it will be optimal to set the next state equal to k_j.

6. Let's now go to the main part of the program. This embeds a loop that computes for each iteration the next update of the Bellman operator, i.e. given v, it computes Tv, until we have that:

$$\|TV - V\| = \max_i |TV[i] - V[i]| < \varepsilon.$$

   We program this as a while loop that iterates until this condition is satisfied.

7. Inside the loop we first have to update the value of V to TV. Notice that in order to do this, we first need to assign the value of TV to V ($V \equiv TV$).

8. Next, we need to compute the new values of $TV[i] = Tv(k_i)$ for all states k_i in the grid. In order to do this, we iterate through the values $i = 1, \ldots N$ and compute each time the values of $TV[i]$ and $G[i]$. This can be done using a For-loop.

   Given a particular value k_i (i.e. in the i-th iteration of the For-loop), we construct the values of the function,

$$f(k') = \begin{cases} \frac{(k_i^\alpha + (1-\delta)k_i - k')^{1-\sigma} - 1}{1-\sigma} + \beta v(k') & \text{if } k' \leq k_i^\alpha + (1-\delta)k_i, \\ -C & \text{if } k' > k_i^\alpha + (1-\delta)k_i \end{cases}$$

   where C is a very big number. This can be done by constructing an N-dimensional vector $F_i = [f(k_i), \ldots, f(k_N)]$.

9. Next, the aim is to find the maximal element of the vector F. This maximal element will be the value $TV[i] = \max_j F_i[j]$. The index j at which this element is found, will be the new value of $G[i]$.

10. Don't forget to close the for and while loops.

Try to count the number of iterations that the program need in order to converge (i.e. the number of iterations of the While-loop) and the time it takes to converge.

Figures 1, 2 and 3 give a plot of the policy function, the value function and the optimal level of consumption as functions of k.

The convergence rate of the Bellman operator has a rate of β. For many economic models, it is natural to choose β close to 1. Convergence of value function iteration method is particularly slow if β is chosen to be close to one.

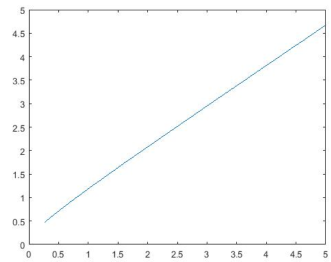

Figure 1: Present capital stock versus next periods capital stock.

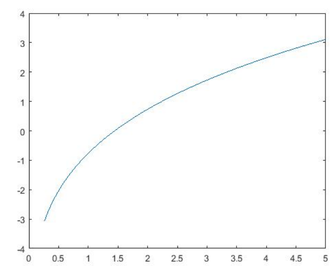

Figure 2: Value function.

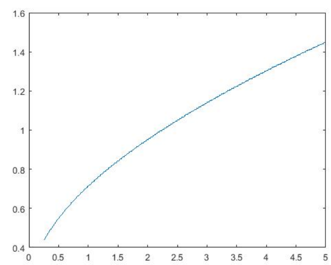

Figure 3: Consumption as a function of capital.

Interpolation

The speed of the value function iteration depends on the size of the grid X. The larger X the more values of $Tv(x)$ we need to compute, and each involves an optimization procedure. A first possible improvement for the speed of the algorithm is to decrease the size of this grid. However, we still would like to have a reasonable good estimate of the value of $Tv(k)$:

$$(Tv)(x) = \max_{a \in \Gamma(x)} \{F(x,a) + \beta v(r(x,a))\}.$$

Keeping x fixed, the quality of this estimate will depend on the grid size of A. The problem, however, is to obtain the value $v(r(x,a))$ on the right hand side. If v is only known on the finite grid X. In other words, it might be the case that $r(x,a)$ takes on a value that is not in this grid, which means that we cannot evaluate $v(r(x,a))$ for this particular value of a and x.

In our example, we had the Bellman operator:

$$(Tv)(k) = \max_{k' \leq k^\alpha + (1-\delta)k} \frac{(k^\alpha + (1-\delta)k - k')^{1-\sigma} - 1}{1-\sigma} + \beta v(k').$$

Here, we have a small grid K but we would like to perform the maximization using a finer grid for k'. As such, we need to evaluate $v(k')$ also for values $k' \notin K$. The main change in our algorithm will therefore take place in step 8 of our algorithm. The function,

$$f(k') = \begin{cases} \frac{(k_i^\alpha + (1-\delta)k_i - k')^{1-\sigma} - 1}{1-\sigma} + \beta \tilde{v}(k') & \text{if } k' \leq k_i^\alpha + (1-\delta)k_i, \\ -C & \text{if } k' > k_i^\alpha + (1-\delta)k_i \end{cases}$$

is now computed on a denser grid. As such, the dimension for f will be larger than N. The function $\tilde{v}(k')$ is computed using interpolation of v on the N-dimensional vector v. As an example, let us put the grid for the k equal to 20 and let the denser grid be equal to 1000. Adjust the code as indicated above using, for example, linear interpolation. After how many rounds does the code converge? What is the time of convergence?

The increase in speed is maily due to the large decrease of the grid size. Unfortunately, the value function is now only known at a smaller number of points and the interpolation function might be a bad guess for true value function. It is also not possible to prove convergence of the algorithm and convergence might even fail if, for example, the interpolation function is not well chosen.

Howard improvement (policy iteration)

THE MOST IMPORTANT FACTOR THAT DETERMINES the speed of the value function iteration algorithm is the optimization routine. Optimization is costly. Therefore, computational improvements should be aimed at reducing the number of times the optimization routine is called. This is the idea behind the Howard improvement algorithm. Let H be the set of all potential policy functions:

$$H = \{h : X \to A : g(x) \in \Gamma(x)\}.$$

For any $g \in H$, we can define an operator R_g such that,

$$(R_g v)(x) = F(x, g(x)) + \beta v(r(x, g(x))).$$

This operator determines the value function resulting from using g as the choice variable. It is easily verified that the operator R_g from $B(X)$ to $B(X)$ satisfies the conditions of Blackwell's theorem so it has a fixed point which can be obtained by iteration. This fixed point satisfies the condition:

$$v(x) = F(x, g(x)) + \beta v(r(x, g(x)))$$

which means that it computes the value of the infinite horizon problem under the constraint that the policy function g is used in every period. Importantly, the computation of this fixed point does not require any optimization routine, so it should be quickly to compute.

The Howard improvement procedure takes the following form.

1. Decide on a grid X and A.
2. Pick any value function v_0.
3. initiate the loop at $t = 1$ for all t, do the following
   (a) Find the policy function g_t such that,
   $$g_t(x) = \arg\max_{a \in \Gamma(x)} F(x, a) + \beta v_{t-1}(r(x, a)).$$
   (b) find v_t as the unique fixed point of R_{g_t}, i.e.
   $$v_t(x) = F(x, g_t(x)) + \beta v_t(r(x, g_t(x))).$$
   (c) iterate steps (a) and (b) each time updating $t \leftarrow t + 1$, until convergence is met: $\|v_t - v_{t-1}\| < \epsilon$.

The Howard algorithm first converges on the value function given the policy function g_t. Once this function is found, the policy function g_t is updated using a maximization step. The advantage of this algorithm is that it requires fewer optimization iterations. Given that this is the most costly step, the algorithm is usually (much) faster.

The following theorem shows the validity of the Howard improvement algorithm.

Theorem 13. *The sequence of functions $(v_t)_{t\in\mathbb{N}}$ of the Howard algorithm converges to the fixed point of the Bellman operator T.*

Proof. Let T be the Bellman operator and let R_g be the policy function iterator for a given a policy function $g \in H$. Let v_n be the policy function obtained by the nth step of the algorithm. We will show that

$$v_0 \leq Tv_0 \leq v_1 \leq Tv_1 \leq \dots$$

This is an increasing sequence in a bounded set,[46], so this sequence converges to the value $\sup_t v_t$ which is a fixed point of the Bellman operator T.

[46] Notice that X is a finite grid, so the number of distinct policy functions is finite.

Let us first show that for all t, $Tv_t \geq v_t$. Indeed,

$$\begin{aligned}(Tv_t)(x) &= \max_{a\in\Gamma(x)} F(x,a) + \beta v_t(r(x,a)),\\ &\geq F(x, g_t(x)) + \beta v_t(r(x, g_t(x))) = v_t(x).\end{aligned}$$

The last equality follows from the fact that v_t is a fixed point of the operator R_{g_t}, so:

$$v_t(x) = F(x, g_t(x)) + \beta v_t(r(x, g_t(x))).$$

Next, we can show that $(Tv_t) = (R_{g_{t+1}} v_t)$. Indeed, by definition of g_{t+1}, we have:

$$(Tv_t)(x) = F(x, g_{t+1}(x)) + \beta v_t(r(x, g_{t+1}(x))) = (R_{g_{t+1}} v_t)(x).$$

Then, if we iterate $R_{g_{t+1}}$ a second time, we get:

$$\begin{aligned}(R^2_{g_{t+1}} v_t)(x) - (R_{g_{t+1}} v_t)(x) =& F(x, g_{t+1}(x)) + \beta (R_{g_{t+1}} v_t)(r(x, g_{t+1}(x))),\\ &- F(x, g_{t+1}(x)) - \beta v_t(r(x, g_{t+1}(x))),\\ &= \beta\left[(Tv_t)(r(x, g_{t+1}(x))) - v_t(r(x, g_{t+1}(x)))\right] \geq 0.\end{aligned}$$

where the last inequality follows from the fact that $Tv_t \geq v_t$. This shows that $(R^2_{g_{t+1}} v_t) \geq R_{g_{t+1}} v_t = Tv_t$.

Now let v_{t+1} be the fixed point of $(R_{g_{t+1}} v_t)$. We will show that $v_{t+1} \geq Tv_t$. In order to do this, we show that $(R^m_{g_{n+1}} v_t) \geq (R^{m-1}_{g_{n+1}} v_t)$ for all $m \geq 2$. As v_{t+1} is the limit of $(R^m_{g_{t+1}} v_t)$ for m going to infinity, this proves the assertion.

For $m = 2$, the proof is given above. Now for the induction step, we have:

$$\begin{aligned}&(R^m_{g_{t+1}} v_t)(x) \geq (R^{m-1}_{g_{t+1}} v_t)(x),\\ \leftrightarrow& F(x, g_{t+1}(x)) + \beta (R^{m-1}_{g_{t+1}} v_t)(r(x, g_{t+1}(x))) \geq F(x, g_{t+1}(x)) + \beta (R^{m-2}_{g_{t+1}} v_t)(r(x, g_{t+1}(x)),\\ \leftrightarrow& (R^{m-1}_{g_{t+1}} v_t)(r(x, g_{t+1}(x))) \geq (R^{m-2}_{g_{t+1}} v_t)(r(x, g_{t+1}(x))).\end{aligned}$$

which is indeed true by the induction hypothesis. Given that $v_{t+1} = \lim_m (R^m_{g_{t+1}} v_t)$, we have that

$$v_{t+1} \geq (R_{g_{t+1}} v_t) = T v_t,$$

as was to be shown. □

The Howard algorithm can be implemented by using a call to a new function that computes the fixed point of the policy function mapping after each optimization routine of the value function iteration (i.e. after each For-loop iteration).

In order to compute the fixed point of the policy function, one can take the following steps.

1. Let the policy function G and the value function TV be the output of the maximization step of the value function iteration.

2. Initialize vectors W and $RW = V$

3. Do the following until $\|W - RW\| < \varepsilon$.

4. assign $W = RW$ and compute the updated value for RW:

$$RW[i] = \frac{(k_i^\alpha + (1-\delta)K[i] - K[G[i]])^{1-\sigma} - 1}{1-\sigma} + \beta W[G[i]].$$

   Here we use index notation, where $K[G[i]]$ uses the index in $G[i]$ to get to the element $G[i] = j$ whenever $g(k_i) = k_j$. The same goes for $W[G[i]]$.

5. Close the while loop

6. Assign the updated value $TV = W$.

Compute the number of times the outer value function While-loop iterates until convergence and compute the time it takes to converge.

Instead of using a loop to compute the fixed point of R_g, it is sometimes possible to explicitly solve this step. Observe that the fixed point of the operator R_g satisfies,

$$v(k) = F(k, g(k)) + \beta v(g(k)).$$

This can be written in vector notation as,

$$V = F(K, K[G]) + \beta Q V.$$

where Q is an $N \times N$ matrix with a 1 at position i, j if and only if $G[i] = 1$. This system can be solved for V,

$$V = (I - \beta Q)^{-1} F(K, K[G]).$$

This necessitates the inversion of the matrix $I - \beta Q$, which is computationally also costly (especially if the size of the grid is large). So it is not always the case that this gives a more efficient way of computing the fixed point of R_g.

Try to code the policy function iteration in this alternative way. For this, you first need to compute the matrix Q and invert $(I - \beta Q)$.

Some applications

Let us have a look at some applications of dynamic programming under certainty.

Optimal tree growth

Consider a tree whose growth is described by a function $h : \mathbb{R}_+ \to \mathbb{R}_+$. In particular, if k_t is the length of the tree in period t then $k_{t+1} = h(k_t)$ is the height of the tree tomorrow. Assume that the price of wood is one per meter of tree, and the interest rate r are both constant over time. Set $\beta = 1/(1+r)$. It is costless to cut down the tree.

If the tree cannot be replanted, the problem in each period is either to cut the tree or not. If the tree is cut in period t then the value is given by $v(k_t) = k_t$ and there is no value thereafter. If the tree is not cut in period t then the value is given by $v(k_t) = \beta v(h(k_t))$. Each period, the problem is either to cut the tree or not. As such,

$$v(k_t) = \max_{c=\{0,1\}} \{k_t c + (1-c)\beta v(h(k_t))\}.$$

Here c is a binary variable that decides whether to cut the tree or not. Observe that his problem can be rewritten as,

$$v(k_t) = \max\{k_t; \beta v(h(k_t))\}.$$

The first choice is taken when the tree is cut while if the second option is take the tree is not cut. Assume that there is a maximum height that the tree can take, $k \in [0, H]$.

Theorem 14. *The operator* $(Tv)(k) = \max\{k, \beta v(h(k))\}$ *is a contraction mapping from the set of bounded functions* $B([0, H])$ *to* $B([0, H])$.

Proof. We check Blackwell's theorem. If $v \leq w$ then

$$(Tv)(k) = \max\{k, \beta v(h(k))\} \leq \max\{k, \beta w(h(k))\} = (Tw)(k),$$

which shows monotonicity. For additivity,

$$\begin{aligned}(Tv+a)(k) &= \max\{k, \beta(v+a)(h(k))\},\\ &= \max\{k, \beta v(h(k)) + \beta a\},\\ &\leq \max\{k+\beta a, \beta v(h(k)) + \beta a\} = (Tv)(k) + \beta a.\end{aligned}$$

□

As such, we know that T has a fixed point. In order to get an idea of the shape of v, we start by a simulation. We set $H = 15$ and consider a grid of fifteen values of $k = 1, 2, \ldots, 15$. We specify $h(k) = k + 0.25(H - k)$ so every period the growth of the tree equals one fourth of the distance between its height and the maximal height.[47] The value function is given in Figure 4. We see that for low values of k, $v(k)$ is above the diagonal, which means that the tree will not be cut.

$$v(k) > k.$$

[47] I don't know if this is realistic but it leads to a concave growth path.

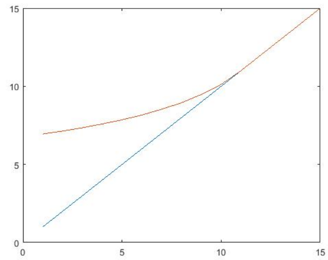

Figure 4: Value function and main diagonal.

For high values of k, we have that $v(k) = k$, which means that the tree will be cut. This indicates that there probably is a unique cutoff height k^* that determines the minimal height for the tree to be cut.

At k^*, the decision maker should then be indifferent between cutting the tree or not. As such, k^* should satisfy the condition:

$$k^* = \beta v(h(k^*)).$$

In addition $h(k^*) \geq k^*$ so we know that (if we adhere to the conjecture), for height $k' = h(k^*)$, the tree will also be cut, i.e. $v(k') = k'$. As such:

$$\begin{aligned}k^* &= \beta v(h(k^*)) = \beta h(k^*),\\ \rightarrow \frac{h(k^*)}{k^*} &= 1/\beta.\end{aligned}$$

The left hand side gives the proportional growth of a tree of height k^*. The right hand side gives the interest rate $(1 + r) = 1/\beta$, i.e. the cost of waiting. If the left hand side is greater than the right hand side, it will be optimal not to cut the tree. Otherwise, cutting is optimal.

The following puts an assumption on the function $h(k)$ that guarantees that this reasoning is correct:

Assumption 1. *Assume that there is a unique $k^* \in [0, H]$ such that,*

- *if $k > k^*$ then $\frac{h(k)}{k} < \frac{1}{\beta}$*
- *if $k < k^*$ then $\frac{h(k)}{k} > \frac{1}{\beta}$.*

Theorem 15. *If assumption 1 is satisfied, then it is optimal to cut the tree for all $k \geq k^*$.*

Proof. Consider the fixed point v^* of the Bellman operator. We need to show that $v^*(k) = k$ whenever $k \geq k^*$, i.e. it is optimal to cut the tree if $k \geq k^*$. First, notice that the Bellman operator T with

$$(Tv) = \max\{k, \beta v(h(k))\}.$$

is a contraction mapping. Let

$$C = \{v \in B([0, H]) : \forall k \geq k^*, v(k) = k\}.$$

Let us first show that C is a closed set. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in C and $v_n \overset{n}{\to} v$. Now, if $k \geq k^*$ then for all n, $v_n(k) = k$, so by taking limits: $v(k) = k$.

As such, if we can show that $T(C) \subseteq C$, we know that the fixed point of T is in the set C, so v^* satisfies the desired condition.

Let $v \in C$. We need to show that $Tv \in C$ or equivalently, for all $k \geq k^*$, $(Tv)(k) = k$. Assume that $k \geq k^*$. Then,

$$(Tv)(k) = \max\{k, \beta v(h(k))\}.$$

we know that $h(k) \geq k \geq k^*$, and $v \in C$, so $v(h(k)) = h(k)$. This gives,

$$(Tv)(k) = \max\{k, \beta h(k)\}.$$

As $k \geq k^*$ we also know that, by assumption, $\frac{h(k)}{k} < 1/\beta$, so,

$$(Tv)(k) = \max\{k, \beta h(k)\} = k.$$

As such, $Tv \in C$ as was to be shown. ☐

The next result states that it is not optimal to cut the tree if $k < k^*$.

Theorem 16. *If assumption 1 is satisfied, then for all $k < k^*$ it is better to wait.*

Proof. As before let v^* be the fixed point of the Bellman operator. We need to show that for $k < k^*$, $\beta v^*(h(k)) > k$.

Let us first show that the fixed point v^* is a non-decreasing function: if $k \geq k'$ then $v(k) \geq v(k')$. Let $C = \{v \in B([0, H]) : v \text{ is non-decreasing}\}$. This is a closed set. As such, in order to show that v^* is non-decreasing, it suffices to show that $T(C) \subseteq C$. Towards this end, let $v \in C$, i.e. v is non-decreasing. Then, if $k \geq k'$,

$$(Tv)(k) = \max\{k, \beta v(h(k))\} \geq \max\{k', \beta v(h(k'))\} = (Tv)(k').$$

which establishes the proof: $v^* \in C$.

Let

$$D = \{v \in B([0, H]) : v \text{ is nondecreasing and } \forall k \leq k^*, \beta v(h(k)) \geq k\}.$$

and let

$$D' = \{v \in B([0,H]) : f \text{ is nondecreasing and } \forall k < k^* : \beta v(h(k)) > k\}.$$

The set D is clearly closed. As such, if we can show that $T(D) \subseteq D'$, we know that the fixed point v^* should be in the set D' which we wanted to show.

So let $v \in D$ (i.e. v is non-decreasing and for all $k \leq k^*$, $\beta v(h(k)) \geq k$) then we need to show that (Tv) is non-decreasing and $k < k^*$ implies $\beta(Tv)((h(k)) > k$. Above, we already showed that T maps non-decreasing functions to non-decrasing functions. For the second part, let $k < k^*$. We need to show that $\beta(Tv)(h(k)) > k$. Now,

$$(Tv)(h(k)) = \max\{h(k), \beta v(h(h(k)))\} \geq \max\{h(k), \beta v(h(k))\}.$$

The inequality follows from the fact that v is a non-decreasing function, so $h(h(k)) \geq h(k)$ implies $v(h(h(k))) \geq v(h(k))$. As $v \in D$, we also know that $\beta v(h(k)) \geq k$, so

$$(Tv)(h(k)) \geq \max\{h(k), \beta v(h(k))\} \geq \max\{h(k), k\} = h(k).$$

Finally given that $k < k^*$, by assumption 1 we know that $\frac{h(k)}{k} > 1/\beta$, so

$$(Tv)(h(k)) \geq h(k) > k/\beta.$$

which is equivalent to $\beta(Tv)(h(k)) > k$, so $(Tv) \in D'$.

We conclude that $v^* \in D'$. □

Above two results show that if assumption 1 is satisfied. Then there is a unique $k^*(= h(k^*)/\beta)$ such that for all $k < k^*$ the tree is not cut and for all $k \geq k^*$ the tree will be cut.

Optimal policy business cycles

The effectiveness of monetary economic policy depends on the expectations of the agents in the economy. Assume that the deviation of y_t which is the log of output from its natural level y^* is given by the following Philips curve:

This model is borrowed from Ginsburgh and Michel, 1998, Optimal policy business cycles, Journal of Economic Dynamics and Control.

$$(y_t - y^*) = \gamma(\pi_t - \pi_t^e),$$

where π_t and π_t^e are the actual and expected rate on inflation in period t. Here $\gamma > 0$. This claims that only unexpected inflation can push output above its natural level. The policy maker's objective in each period is given by a trade off between more output and less inflation:

$$g(y_t, \pi_t) = \alpha(y_t - y^*) - \frac{\pi_t^2}{2} = \alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2}.$$

The forward looking government has a discount factor δ so the problem is to maximize:

$$\sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2} \right).$$

If agents have rational expectations then expected inflation equals actual inflation, so $\pi_t^e = \pi_t$ and therefore $y_t = y^*$. In this case, the optimal policy is to set $\pi_t = 0$ at every point in time. Now, assume that not all agents have rational expectations. Some agents have adaptive expectations in the sense that:[48]

$$\pi_{t+1}^a = \lambda\pi_t + (1-\lambda)\pi_t^a,$$

where $\lambda \in (0, 1]$. Assume that a proportion x_t of agents use rational expectations while a fraction $(1 - x_t)$ form adaptive expectations. We assume that the average expected rate of inflation is a weighted average of the rates expected by rational and adaptive agents:

$$\pi_t^e = x\pi_t + (1-x)\pi_t^a.$$

then,

$$\begin{aligned} &\sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2} \right), \\ &= \sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma(1-x)(\pi_t - \pi_t^a) - \delta^t \frac{\pi_t^2}{2} \right). \end{aligned}$$

The government will try to set π_t such as to maximize this payoff. The Bellman equation is:

$$v(\pi_t^a) = \max_{\pi} \left\{ \alpha\gamma((1-x)(\pi_t - \pi_t^a) - \frac{\pi_t^2}{2} + \delta v(\lambda\pi_t + (1-\lambda)\pi_t^a) \right\}.$$

Let us try to derive the Euler equations. Let $q_t = v'(\pi_t^a)$ then the first order condition and envelope theorem give:

$$\begin{aligned} q_t &= -\alpha\gamma(1-x) + \delta(1-\lambda)q_{t+1}, \\ 0 &= \alpha\gamma(1-x) - \pi_t + \delta\lambda q_{t+1}. \end{aligned}$$

So, eliminating the q_t, q_{t+1} variables gives:

$$\begin{aligned} &\frac{\pi_{t-1} - \alpha\gamma(1-x)}{\delta\lambda} = -\alpha\gamma(1-x) + \delta(1-\lambda)\frac{\pi_t - \alpha\gamma(1-x)}{\delta\lambda}, \\ &\leftrightarrow \pi_{t-1} - \alpha\gamma(1-x) = -\alpha\gamma(1-x)\delta\lambda + \delta(1-\lambda)\pi_t - \alpha\gamma(1-x)\delta(1-\lambda), \\ &\leftrightarrow \pi_{t-1} - \delta(1-\lambda)\pi_t = \alpha\gamma(1-x)(1-\delta), \\ &\leftrightarrow \pi_t - \frac{\pi_{t-1}}{\delta(1-\lambda)} = -\frac{\alpha\gamma(1-x)(1-\delta)}{\delta(1-\lambda)}. \end{aligned}$$

[48] For example, they form expectations that are adaptive.

This is an explosive difference equation, so the only solution is the one at the steady state, where

$$\pi^* = \frac{\alpha\gamma(1-x)(1-\delta)}{1-\delta(1-\lambda)}.$$

NOW LET US ENDOGENEIZE the share of rational agents x_t. Assume that at time t decisions are made at no cost on the basis of adaptive expectations π_t^a. An agent θ can modify this decision at a fixed cost c using the new information π_t. There is a continuum of agents $\theta \in [0,1]$. Agent θ makes a cost equal to $\theta(\pi_t^a - \pi_t)^2$ when he uses π_t^a instead of π_t. let $\underline{\theta}_t$ be defined by,

$$\underline{\theta}_t(\pi_t^a - \pi_t)^2 = c.$$

The loss of agent θ is larger than c if $\theta \geq \underline{\theta}_t$. and the proportion x_t of agents that decide to change their decision is,

$$x_t = x(\pi_t^a, \pi_t) = \max\{0, 1 - c(\pi_t^a - \pi_t)^{-2}\}.$$

The Bellman equation is now,

$$v(\pi_t^a) = \max_{\pi_t}\left\{\beta(1-x_t)(\pi_t - \pi_t^a) - \frac{\pi_t^2}{2} + \delta v(\lambda\pi_t + (1-\lambda)\pi_t^a)\right\},$$
$$\text{s.t. } x_t = \max\{0, 1 - c(\pi^a - \pi_t)^{-2}\}.$$

The model is a bit daunting to analyze analytically, so we will resort to a simulation exercise. We use parameter values $\beta = 0.1$, $c = 0.0001$, $\lambda = 0.75$ and $\delta = 0.95$. Also, we use grid of 1000 values of π^a between -0.1 and 0.05. Figure 5 plots π_{t+1}^a against the value of π_t^a. The stable state is situated at the point where the curve intersects with the diagonal. One sees that below the steady state the best response is above the diagonal. So, π^a increases over time. Suddenly the best response drops to below the diagonal. This shows that π^a will show cyclical behaviour. It will gradually increase and then suddenly drop to a lower value after which it will start increasing again.

Figure 6 shows a the evolution of inflation over time. Here the cyclical behaviour is clearly visible. For this example, we have cycles of length 6. In 5 periods, inflation increases stepwise. In the sixth period inflation drops again to its starting value.

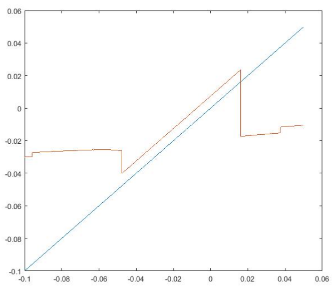

Figure 5: Value of π_{t+1}^a against the value of π_t^a.

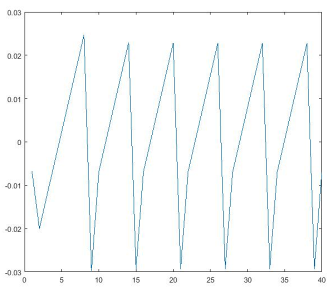

Figure 6: Evolution of π over time.

Stochastic dynamic programming

Intuitively, a stochastic dynamic program has the same components as a deterministic one. The only (major) difference is that the transition that governs the process of going from one state to another is no longer certain. When transitions occur probabilistically, states and decisions today lead to a distribution over possible states in the future.

As before, we use X to be the set of states and let A be the set of actions. Also similar as before, we define a correspondence $\Gamma : X \to A$ that determines which actions the agent can take for a given state $x \in X$. In the deterministic case, the state in the next period was given by the function $r(x, a)$. This is no longer the case in the stochastic world. Instead of the transition function we now introduce a Markov transition kernel.

$$Q((x, a), B),$$

Here $Q((x, a), B)$ gives the probability that the state in the next period is in the set $B \subseteq X$ if the state today is x and the action taken today is a.[49] Notice that the probability law of the state tomorrow only depends on the state and action taken today. It does not depend on what happened before today. A process that does not depend on the past, given the present is called a Markov process.

[49] Formally, $Q((x, a), B)$ is a kernel if $Q((x, a), .)$ is a probability measure for all (x, a) and $Q(., B)$ is a measurable function for all measurable sets B.

If $f : X \to \mathbb{R}$ is a payoff function then the expected value of f tomorrow if (x, a) is the current state and $a \in \Gamma(x)$ is chosen is given by:

$$\int_X Q((x, a), dx') f(x')$$

As before, a policy function $g : X \to A$ determines for each state X an action $a \in A$ taken by the decision maker. Assume that the current state is x_0. Then if the decision maker follows the policy rule g, the next period's expected payoff is given by,

$$E_0[\beta F(x_1, g(x_1)|x_0] = \int_X Q((x_0, g(x_0)), dx_1) \beta F(x_1, g(x_1)).$$

The expected payoff within two periods F two periods from now is given by,

$$\begin{aligned} & E_0[\beta^2 F(x_2, g(x_2)|x_0], \\ & = \int_X Q((x_0, g(x_0)), dx_1) \int_X Q((x_1, g(x_1)), dx_2)\beta^2 F(x_2, g(x_2)). \end{aligned}$$

The expected payoff for the first n periods is then determined by,

$$\begin{aligned} u_n(g) &= E_0\left[\sum_{t=1}^{n} \beta^t F(x_t, g(x_t)) \middle| x_0\right], \\ &= F(x_0, g(x_0) + \int_X Q((x_0, g(x_0)), dx_1)\beta F(x_1, g(x_1), \\ &+ \int_X Q((x_0, g(x_0)), dx_1) \int_X Q((x_1, g(x_1)), dx_2)\beta^2 F(x_2, g(x_2)). \\ &+ \dots, \\ &+ \int_x Q((x_0, g(x_0), dx_1) \int_X Q((x_1, g(x_1)), dx_2) \int_X \dots \int_X Q((x_{n-1}, g(x_{n-1})), dx_n)\beta^n F(x_n, g(x_n). \end{aligned}$$

Let $u_\infty(g) = \lim_n u_n(g)$, if it exists. The aim of the decision maker is to find a policy function g to maximize $u_\infty(g)$,[50]

$$\sup_g u_\infty(g).$$

[50] Observe that we have not showed yet that this maximization problem is well defined.

The aim of this chapter is to relate the solution of this problem (if it exists) to the solution of the following functional equation,

$$\begin{aligned} v(x) &= \max_{a\in\Gamma(x)} \left\{F(x, a) + \beta \int_X Q((x, a), dx')v(x')\right\}, \\ &= \max_{a\in\Gamma(\omega)} \left\{F(x, a) + \mathbb{E}\left[v(x')|(x, a)\right]\right\} \end{aligned}$$

This is the Bellman equation for the stochastic problem. It is related to the following Bellman operator T,

$$(Tv)(x) = \max_{a\in\Gamma(\omega)} \left\{F(x, a) + \beta \int_X Q((x, a), dx')v(x')\right\},$$

Definition 13 (regularity)**.** *The stochastic dynamic programming problem $(X, A, \Gamma, F, \beta, Q)$ is regular if the instantaneous payoff function $F : X \times A \to \mathbb{R}$ is continuous, the transition function $\Gamma : \Omega \to X$ is non-empty and continuous and there exists a continuous function $\phi : \Omega \to \mathbb{R}_{++}$ such that,*

1. *There exists an $M \geq 0$ such that for all $x \in X$,*

$$\max_{a\in\Gamma(x)} |F(x, a)| \leq M\phi(\omega).$$

2. *There exists a* $\theta \in (0,1)$ *such that for all* $x \in X$,

$$\beta \max_{a\in\Gamma(x)} \int_X Q((x,a),dx')\phi(x') \leq \theta\phi(\omega).$$

3. *for any* $(x,a) \in X \times A$,

$$\int_X Q((x,a),dx')\phi(x') < \infty.$$

4. *If* $f : X \to \mathbb{R}$ *is continuous and* $f \in B_\phi(X)$*, then*

$$R(x,a) = \int_X Q((x,a),dx')f(x'),$$

is a bounded continuous function on $X \times A$.

Condition 4 is called the Feller condition.

Theorem 17. *If the problem* $(X, A, \Gamma, F, \beta, Q)$ *is regular then the Bellman operator maps* $B_\phi(X)$ *into* $B_\phi(X)$ *and is a contraction mapping.*

Proof. Let $v \in B_\phi(X)$ then, there is an $M > 0$ such that,

$$\begin{aligned}\int_X Q((x,a),dx')|v(x')| &\leq \int_X Q((x,a),dx')M\phi(\omega'),\\ &\leq \infty\end{aligned}$$

This shows that v is integrable. Let us show that T maps $B_\phi(X)$ into $B_\phi(X)$. If $v \in B_\phi(X)$ then Tv is continuous by the theorem of the maximum. To see that (Tv) is bounded in the $\|.\|_\phi$ norm, observe that,

$$\begin{aligned}|(Tv)(X)| &= \left|\max_{a\in\Gamma(x)}\left\{F(x,a) + \beta\int_X Q((x,a),dx')v(x')\right\}\right|,\\ &\leq \max_{a\in\Gamma(x)}|F(x,a)| + \beta\max_{a\in\Gamma(x)}\int_X Q((x,a),dx')|v(x')|,\\ &\leq \max_{a\in\Gamma(x)}|F(x,a)| + \beta\max_{a\in\Gamma(x)}\int_X Q((x,a),dx')\|v\|_\phi\phi(x'),\\ &\leq M\phi(x) + \beta\|v\|_\phi\theta\phi(x),\\ &= (M + \|v\|_\phi\theta)\phi(x).\end{aligned}$$

so $\|(Tv)\|_\phi$ is bounded by $M + \|v\|_\phi\theta$ which is finite.

For a contraction mapping, we verify Blackwell's conditions. For monotonicity, let $v \geq w$ then

$$\begin{aligned}(Tv)(x) &= \max_{a\in\Gamma(x)}\left\{F(x,a) + \beta\int_X Q((x,a),dx')v(x')\right\},\\ &\geq \max_{a\in\Gamma(x)}\left\{F(x,a) + \beta\int_X Q((x,a),dx')w(x')\right\},\\ &= (Tw)(x).\end{aligned}$$

For additivity,

$$
\begin{aligned}
(T(v+\alpha\phi))(x) &= \max_{a\in\Gamma(\omega)} \left\{ F(x,a) + \beta \int_X Q((x,a),dx')(v+\alpha\phi)(x') \right\}, \\
&\leq (Tv)(x) + \beta\alpha \max_{a\in\Gamma(x)} \int_X Q((x,a),dx')\phi(x'), \\
&\leq (Tv)(\omega) + \theta\alpha\phi(x),
\end{aligned}
$$

as was to be shown. □

Now, let's go back to our original problem,

$$
\sup_g u_\infty(g).
$$

We will relate the solution to this problem with the fixed point of the Bellman operator.

Lemma 7. *Let $(X, A, F, \Gamma, \beta, Q)$ be a regular problem. Let g be a policy function. Then $u_\infty(g)$ exists and the set $\{u_\infty(g) : g : X \to A\}$ is bounded from above, so the* sup *problem is well defined.*

Proof. For a given $g : X \to A$, we have that,

$$
\begin{aligned}
u_n(g) = {} & F(x_0, g(x_0)), \\
& + \beta \int_X Q((x_0, g(x_0)), dx_1) F(x_1, g(x_1)), \\
& + \beta \int_X Q((x_0, g(x_0)), dx_1) \int_X Q((x_1, g(x_1)), dx_2) F(x_2, g(x_2), \\
& + \ldots, \\
& + \beta^n \int_X Q((x_0, g(x_0)), dx_1) \ldots \int_X Q((x_{n-1}, g(x_{n-1})), dx_n) F(x_n, g(x_n)).
\end{aligned}
$$

Take a term in this summation and take the innermost integral,

$$
\begin{aligned}
\int_X Q((x_{n-1}, g(x_{n-1}), dx_n) F(x_n, g(x_n)) &\leq \int_X Q((x_{n-1}, g(x_{n-1})), dx_n) |F(x_n, g(x_n))|, \\
&\leq \int_X Q((x_{n-1}, g(x_{n-1})), dx_n) M\phi(x_n), \\
&\leq M\frac{\theta}{\beta}\phi(x_{n-1})
\end{aligned}
$$

Then taking the two inner integrals,

$$
\begin{aligned}
& \int_X Q((x_{n-2}, g(x_{n-2})), dx_{n-1}) \int_X Q((x_{n-1}, g(x_{n-1})), dx_n) F(x_n, g(x_n)), \\
& \leq \int_X Q((x_{n-2}, g(x_{n-2})), dx_{n-1}) M\frac{\theta}{\beta}\phi(x_{n-1}), \\
& \leq M\frac{\theta^2}{\beta^2}\phi(x_{n-2}).
\end{aligned}
$$

Iterating further gives that the $t-$th term is bounded from above by $\beta^t M \frac{\theta^t}{\beta^t}\phi(x_0)$. Doing this for every term gives,

$$\begin{aligned} u_n(g) &\leq M\phi(x_0)(1+\theta+\theta^2+\ldots,+\theta^n), \\ &\leq M\phi(x_0)\frac{1}{1-\theta}. \end{aligned}$$

So $u_n(g)$ is bounded. □

Next, let us show that the fixed point of the Bellman operator is greater than any $u_\infty(h)$.

Lemma 8. *Let $(X, A, F, \Gamma, \beta, Q)$ be a regular problem. Let g be a policy function and let v be the fixed point of the Bellman operator, then $v(x_0) \geq u_\infty(g)$.*

Proof. We have that,

$$\begin{aligned} v(x_0) &\geq F(x_0, g(x_0)) + \beta \int_X Q((x_0, g(x_0), dx_1)v(x_1), \\ &\geq F(x_0, g(x_0)) + \beta \int_X Q((x_0, g(x_0)), dx_1)F(x_1, g(x_1)), \\ &+ \beta^2 \int_X Q((x_0, g(x_0)), dx_1) \int_X Q((x_1, g(x_1)), dx_2)v(x_2), \\ &= \ldots, \\ &= u_n(g) + \beta^{n+1} \int_X Q((x_0, g(x_0)), dx_1)... \int_X Q((x_n, g(x_n)), dx_{n+1})v(x_{n+1}). \end{aligned}$$

Taking the limit to infinity, the first term goes to $u_\infty(g)$. So we only need to show that the second term goes to zero. However, the inner integral is bounded by,

$$\begin{aligned} &\int_X Q((x_n, g(x_n)), dx_{n+1})v(x_{n+1}), \\ &\leq \|v\|_\phi \int_X Q((x_n, g(x_n)), dx_{n+1})\phi(x_{n+1}), \\ &\leq \|v\|_\phi \frac{\theta}{\beta}\phi(x_n) \end{aligned}$$

Iterating further over all other integrations gives finally, that the term is bounded from above by,

$$\beta^{n+1}\|v\|_\phi \frac{\theta^{n+1}}{\beta^{n+1}}\phi(x_0)$$

This goes to zero as $n \to \infty$. □

For $x \in X$ define the value $g(x)$ as,

$$g(x) \in \arg\max_{a\in\Gamma(x)} \left\{ F(x, a) + \beta \int_X Q((x, a), dx')v(x') \right\}.$$

Lemma 9. *Let* $(X, A, F, \Gamma, \beta, Q)$ *be a regular problem, then* $v(x_0) = u_\infty(g)$.

Proof. We have that,

$$\begin{aligned}
v(x_0) &= F(x_0, g(x_0)) + \beta \int_X Q((x_0, g(x_0)), dx_1) v(x_1), \\
&= F(x_0, g(x_0)) + \beta \int_X Q((x_0, g(x_0)), dx_1) F(x_1, g(x_1)), \\
&+ \beta^2 \int_X Q((x_0, g(x_0)), dx_1) \int_X Q((x_1, g(x_1)), dx_2) v(x_2), \\
&= \ldots, \\
&= u_n(g) + \beta^{n+1} \int_X Q((x_0, g(x_0)), dx_1) ... \int_X Q((x_n, g(x_n)), dx_{n+1}) v(x_{n+1}).
\end{aligned}$$

Taking the limit to infinity, the first term goes to $u_\infty(g)$. So we only need to show that the second term goes to zero. However, the inner integral is bounded by,

$$\begin{aligned}
&\int_X Q((x_n, g(x_n)), dx_{n+1}) v(x_{n+1}), \\
&\leq \|v\|_\phi \int_X Q((x_n, g(x_n)), dx_{n+1}) \phi(x_{n+1}), \\
&\leq \|v\|_\phi \frac{\theta}{\beta} \phi(x_n)
\end{aligned}$$

Iterating further over all other integrations gives finally, that the term is bounded from above by,

$$\|v\|_\phi \theta^{n+1} \phi(x_0)$$

This goes to zero as $n \to \infty$. □

Simulations for models of uncertainty

Let us first look at a very simple model of optimal growth with stochastic shocks. We take the utility of the consumer to be $u(c) = \ln(c+1)$. Output is produced using outputs in the previous period net of consumption. In particular, the output in period $t+1$ is given by,

$$y_{t+1} = \eta_{t+1}(y_t - c_t)^\alpha,$$

where η_{t+1} is a stochastic shock with distribution P, realized in period $t+1$. We assume that η_t is i.i.d. The maximization problem reads:

$$\begin{aligned} &\sum_{t=0}^{\infty} \mathbb{E}\left(\beta^t \max_{c_t} \ln(c_t+1) | x_0\right), \\ \text{s.t. } & c_t \leq y_t, \\ & y_{t+1} = \eta_{t+1}(y_t - c_t)^\alpha. \end{aligned}$$

This gives rise to the Bellman equation:

$$v(y) = \max_{c \leq y} \left(\ln(c) + \beta \int_{\mathbb{R}} P(d\eta) v(\eta(y-c)^\alpha)\right).$$

Simulation of this model is analogues as for the case under certainty. The only difference is here to estimate the integral. This can be done using Monte-Carlo simulation.

- Draw a large number of random variables $\eta_1, \ldots, \eta_N$ according to the distribution P.

- Compute the mean,

  $$\frac{1}{N} \sum_{n=1}^{N} v(\eta_n(y_t - c_t)^\alpha).$$

  Here $v(\eta_n(y_t - c_t)^\alpha)$ should be computed by interpolating the values of v.

For the algorithm, it is important to draw the values of $\eta_1, \eta_2, \ldots$ before entering the loop on the function value iteration. This guarantees the convergence of the algorithm. If you draw for each loop new random values, convergence is not guaranteed.

Try to code this problem using Howard improvement assuming that $\eta = e^{\mu + s\varepsilon}$ where ε has a standard normal distribution. You can use the parameters

$$\alpha = 0.4, \qquad \beta = 0.96, \qquad \mu = 0, \qquad s = 0.1$$

Take a grid size of 100 and let the values of y be equally spaced between 0 and 7. Compute the mean based on a sample of 1000 draws of η.

The previous example was rather easy in the sense that the value function (and policy function) where independent of the stochastic component. In particular P did not depend on the state or the action taken. In more interesting examples, however, this is no longer the case. Let's consider a growth model with a representative consumer with utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. Capital is accumulated according to the law of motion:

$$k_{t+1} = e^s k_t^\alpha - c_t + (1-\delta)k_t.$$

Here s is a random variable that takes on two possible values s_1 and s_2. If the state is s_1, then output is multiplied by e^{s_1} if the state is s_2, output is multiplied by s_2. The transition probability between the states over time is determined by a Markov transition matrix:

$$\Pi = \begin{bmatrix} \pi_1 & 1-\pi_1 \\ 1-\pi_2 & \pi_2 \end{bmatrix}.$$

Here π_i is the probability of being in a_i next period, given that a_i is the current state. The optimization problem is then,

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t \mathbb{E}\left(\frac{c_t^{1-\sigma}-1}{1-\sigma} \middle| k_0 \right), \\ \text{s.t. } & k_{t+1} = e^s k_t^\alpha - c_t + (1-\delta)k_t, \\ & \Pr(s_i|s_i) = \pi_i. \end{aligned}$$

A state is now given by a combination of a level of capital k and the value of the shock s. In terms of the Bellman equation, we have:

$$v(k,s) = \max_{c \le e^s k^\alpha + (1-\delta)k} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \beta \sum_{i=1}^{2} \Pr(s_i|s) v(e^{s_i}k^\alpha - c + (1-\delta)k, s_i) \right\}.$$

The value function $v(k,s)$ now depends on two variables, the capital stock k and the shock s. Likewise, the policy function will now be a function that takes a value for k and s and gives a level of consumption, $g(k, s)$. Given that we have two levels for the shocks, we have two functions $v(k, s_1)$ and $v(k, s_2)$ and two policy functions $g(k, s_1)$ and $g(k, s_2)$.

In this sense, we can encode the value function v and policy function g as an $N \times 2$ dimensional vector.

Try to code this problem using a Howard improvement with parameter values,

$$\sigma = 1.5, \qquad \delta = 0.1, \qquad \beta = 0.95, \qquad \alpha = 0.3,$$
$$s_1 = 0.8, \qquad s_2 = 1.2, \qquad \pi_1 = \pi_2 = 0.9,$$

and a grid size of 1000 where k is equally spaced between 0.2 and 6.

Try to use the output of the program to simulate trajectories of the capital stock and trajectories of consumption paths over time.

Applications

Consider the problem of a cake of size x that has to be eaten in its entirety in one single period. There is a taste shock z that takes on two possible values $0 < z_\ell < z_h$. Let $p_{\ell h}$ be the probability of tastesthat the taste sm z_ℓ z_h and let $p_{h\ell}$ be the probability of switching from z_h to z_ℓ. Eating the cake of size x gives a value of $zu(x)$ where $u(x) > 0$ is assumed to be strictly decreasing.

To ad an interesting twist, assume that if the cake is not eaten today, then a fraction $(1 - \delta)$ of the cake is lost. The state of the system depends on the size of the cake and the value of the taste shock. The Bellman equation takes the following expression:

$$v(x, z_\ell) = \max\{z_\ell u(x); \beta[p_{\ell h} v(\delta x, z_h) + (1 - p_{\ell h}) v(\delta x, z_\ell)]\},$$
$$v(x, z_h) = \max\{z_h u(x); \beta[p_{h\ell} v(\delta x, z_\ell) + (1 - p_{h\ell}) v(\delta x, z_h)]\}.$$

It is easy to see that the function $v(x, z)$ should be non-decreasing in x.

Lemma 10. *The fixed point of the Bellman equation satisfies that for all sizes x, $v(x, z_h) \geq v(x, z_\ell)$ and $z_h u(x) \geq \beta v(\delta x, z_h)$. This also means that the cake is always eaten if $z = z_h$.*

Proof. Let,

$$C = \{v : v(x, z_h) \geq v(x, z_\ell) \text{ and } z_h u(x) \geq \beta v(\delta x, z_h)\}.$$

Notice that C is a closed set, so we only need to show that $T(C) \subseteq C$. Let $v \in C$. Then, if $(Tv)(x, z_\ell) = z_\ell u(x)$, we have:

$$(Tv)(x, z_\ell) = z_\ell u(x) < z_h u(x) \leq (Tv)(x, z_h).$$

Next, let $(Tv)(x, z_\ell) > z_\ell u(x)$. Then:

$$(Tv)(x, z_\ell) = \beta p_{\ell h} v(\delta x, z_h) + \beta(1 - p_{\ell h}) v(\delta x, z_\ell),$$
$$\leq \beta v(\delta x, z_h) \leq z_h u(x) \leq (Tv)(x, z_h).$$

This shows the first part of the proof.

Next,

$$\begin{aligned}\beta(Tv)(\delta x, z_h) &= \beta \max\{z_h u(\delta x), \beta p_{h\ell} v(\delta^2 x, z_\ell) + \beta(1-p_{h\ell})v(\delta^2 x, z_h)\},\\ &\leq \beta \max\{z_h u(\delta x), \beta v(\delta^2 x, z_h)\},\\ &\leq \beta \max\{z_h u(\delta x), z_h u(\delta x)\} = \beta z_h u(\delta x) \leq z_h u(x).\end{aligned}$$

Given that the fixed point, v, is in C, we have that:

$$\begin{aligned}z_h u(x) &\geq \beta v(\delta x, z_h),\\ &\geq \beta p_{h\ell} v(\delta x, z_\ell) + \beta(1-p_{h\ell})v(\delta x, z_h).\end{aligned}$$

As such, $v(x, z_h) = z_h u(x)$ which means that the cake will be eaten if $z = z_h$. □

Now, we would like to determine what happens if $z = z_\ell$. In this state, the decision maker faces a trade off between consuming now immediately and getting $z_\ell u(x)$ or waiting one period and hoping that the state changes to $z = z_h$.

Consider the following decision rule. If $x \leq x^*$, eat the cake when $z = z_\ell$. If $x > x^*$, wait for the next period. We would like to know when this is an optimal strategy.

Assume that we are at x^*. In this case, the decision maker should be indifferent between eating and not eating. As such:

$$\begin{aligned}z_\ell u(x^*) &= \beta\left(p_{\ell h} z_h u(\delta x^*) + (1-p_{\ell h}) z_\ell u(\delta x^*)\right),\\ &= \beta u(\delta x^*)\left(z_h p_{\ell h} + z_\ell(1-p_{\ell h})\right),\\ \leftrightarrow \frac{z_\ell}{\beta\left(z_h p_{\ell h} + z_\ell(1-p_{\ell h})\right)} &= \frac{u(\delta x^*)}{u(x^*)}.\end{aligned}$$

where we used the fact that $\delta x \leq x$, so $v(\delta x, z_\ell) = z_\ell u(\delta x)$.

Lemma 11. *Assume that there is an x^* such that:*

$$x \leq x^* \leftrightarrow \frac{z_\ell}{\beta\left(z_h p_{\ell h} + z_\ell(1-p_{\ell h})\right)} \geq \frac{u(\delta x^*)}{u(x^*)}.$$

Then it is optimal to eat at $z = z_\ell$ if and only if $x \leq x^*$.

Proof. Let C be the set such that:

$$C = \{v : \forall x \leq x^*, v(x, z_\ell) = z_\ell u(x)\}.$$

One can show that $T(C) \subseteq C$, so the fixed point of the Bellman equation is in C, which means that at $z = z_\ell$, the decision maker will eat the cake when $x \leq x^*$.

Next, let D be the set such that:

$$D = C \cap \{v : \forall x > x^*, v(x, z_\ell) > z_\ell u(x)\}$$

One can show that $T(C) \subseteq D$, so the fixed point should be in D showing that the cake is not eaten if $x \geq x^*$. □

Now, consider a second possibility, where there exists an x^* such that for $x \leq x^*$ the decision maker waits and for $x > x^*$ the decison maker eats the cake when $z = z_\ell$. Let us denote by $w(x)$ the value of the strategy to always wait if $z = z_\ell$ and eating as soon as $z = z_h$. This is given by:

$$\begin{aligned} w(x) &= \beta(p_{\ell h} z_h u(\delta x)) + \beta(1 - p_{\ell,h}) w(\delta x), \\ &= \beta(p_{\ell h} z_h u(\delta x) + (1 - p_{\ell h}) p_{\ell h} z_h u(\delta^2 x)) + \beta^2 (1 - p_{\ell h})^2 w(\delta^2 x), \\ &= \ldots, \\ &= z_h p_{\ell h} \sum_{t=1}^{n} \beta^t (1 - p_{\ell h})^{t-1} u(\delta^t x) + \beta^t (1 - p_{\ell h})^t w(\delta^t x). \end{aligned}$$

This converges to a finite value.

At x^*, the decision maker should be indifferent between eating and not, so:

$$z_\ell u(x^*) = w(x^*).$$

Lemma 12. *Assume that there is an x^* such that:*

$$x \geq x^* \text{ iff } z_\ell u(x) \geq w(x)$$

then it is optimal to eat at $z = z_\ell$ if and only if $x \geq x^*$.

Optimal stopping problems

Optimal stopping problems are a special class of problems in where the discrete choice is a single decision to put an end to an ongoing problem.[51]

As a first example, consider a burglar who loots a house every day. The daily gains are independent and identically distributed on $\mathbb{R}_+$. With a certain probability $1 - p \in (0, 1)$, the burglar is caught and all her fortune is gone. The utility function of the burglar (with fortune x) is given by $1 - e^{-\alpha x}$.

The Bellman equation takes the form:

$$v(x) = \max \left\{ 1 - e^{-\alpha x}, \beta p \int_{\mathbb{R}_+} v(x + g) P(dg) \right\}$$

where P is the distribution of gains.

Now, assume that there is an x^* such that the Burglar stops for all $x \geq x^*$ and continues for all $x < x^*$. In this case, she should be

[51] For example, a student has to decide when to give up trying to solve a homework problem. A firms decides when to leave an industry, a firm decides when to stop working on the development of a new product or an unemployed worker has to decide when to accept a job from a sequence of offers.

indifferent between stopping and continuing at x^*. This gives:

$$\begin{aligned} 1 - e^{-\alpha x^*} &= \beta p \int_{\mathbb{R}_+} v(x^* + g) P(dg), \\ &= \beta p \int_{\mathbb{R}_+} (1 - e^{-\alpha(x^*+g)}) P(dg), \\ &= \beta p \left(1 - e^{-\alpha x^*} \int_{\mathbb{R}_+} e^{-\alpha g} P(dg) \right), \\ &= \beta p (1 - R e^{-\alpha x^*}) \end{aligned}$$

where $R = \int e^{-\alpha g} P(dg)$ which is a fixed number. So:

$$(1 - \beta p) = (1 - \beta p R) e^{-\alpha x^*}.$$

The right hand side is decreasing in x^*.

Lemma 13. *Let x^* be the value that satisfies*

$$(1 - \beta p) = (1 - \beta p R) e^{-\alpha x^*},$$

then it is optimal to stop if and only if $x \geq x^*$.

Consider an agent that visits stores at a rate of one per period. Then given that the price quoted in the current period is p, the individual can choose to stop now and purchase the good or go to the next store. If he stops, he gets $u - p$ where u is the value of the good bought. If he continuous, he enters the next period as an active searcher. The Bellman equation is,

$$v(p) = \max\{u - p; -\beta c + \beta \int_0^\infty v(p') F(dp')\}.$$

Observe that the second term $-\beta c + \beta \int_0^\infty v(p') F(dp') = \bar{v}$ is independent of the current price p as we assumed that prices are i.i.d. drawn. The first term is declining in p so there is a unique value p^* where $u - p^* = -\beta c + \beta \int_0^\infty v(p') F(dp')$. From this, it follows that $\bar{v} = u - p^*$.

Any price greater than p^* induces further search while any value below p^* let's the agent buy the good. We have that,

$$\begin{aligned} u - p^* &= -\beta c + \beta \int_0^{p^*} v(p') F(dp') + \beta \int_{p^*}^\infty v(p') F(dp'), \\ &= -\beta c + \beta \int_0^{p^*} (u - p') F(dp') + \beta \int_{p^*}^\infty (u - p^*) F(dp'), \\ &= -\beta c + \beta (u - p^*) + \beta \int_0^{p^*} (p' - p^*) F(dp'). \end{aligned}$$

So,

$$p^* = u - \frac{\beta}{1 - \beta} \left[-c + \int_0^{p^*} (p' - p^*) F(dp') \right]$$

This is the fundamental reservation price equation of the problem. The first term gives the immediate benefit of purchasing. The second term gives the option value of waiting.