Trial exam

- The exam lasts 3 hours. There are 8 questions in total so you have on average 22 and a half minute per question.
- If you use a result from the course in order to answer your question make sure that you argue why all assumptions to use the result are valid. For example, if you use Brouwer's fixed point theorem, you need to demonstrate that the domain S is compact and convex and the function maps from S to S and is continuous.

Logic and proofs

1. Let p, q and r be formula's. Show that the formula $(p \to q) \vee r$ is equivalent to the formula $(\neg p \vee q \vee v)$. (i) Derive the truth tables for these formula's. (ii) Use formal rules of deduction to demonstrate the equivalence between the two formula's. Provide for each step in the derivation the specific rule that you apply.

2. Prove that for all integers $n \in \mathbb{N}$ with $n > 3$ and for all numbers $\delta \in \mathbb{R}$ with $\delta \neq 1$.

$$\sum_{t=3}^{n} \delta^t = \frac{\delta^3 - \delta^{n+1}}{1 - \delta}$$

Specify the particular type of proof that you use.

Warming up

1. Let A and B be two non-empty and bounded sets of real numbers, $A, B \subset \mathbb{R}$. Define the set C by,

$$C = A + B = \{x + y \in \mathbb{R} | x \in A, y \in B\}$$

Prove that

$$\sup A + \sup B = \sup C,$$

where sup is the supremum operator, e.g. $\sup A$ is the lowest upperbound of A.

2. Show that the function $f : [-2, 0] \to \mathbb{R}$ where $f(x) = x^3 - x + 1$ has a root in it's domain $[-2, 0]$, i.e. there is a number $x^* \in [-2, 0]$ such that $f(x^*) = 0$.

Fixed points

1. Let C be a correspondence on the set X. For each of the following examples, draw a picture of the correspondence C and verify whether all conditions of Kakutani's fixed point theorem apply.

- $X =]0,1[$ and $C(x) = \{x^2\}$.
- $X = \mathbb{R}_+$ and $C(x) = \{1 + x\}$.
- $X = [0,1]$ and $C(x) = \begin{cases} [1, 2/3] \text{ for } x < 1/3, \\ [2/3, 1] \cup [0, 1/3] \text{ for } x = 1/2, \\ [0, 1/3] \text{ for } x > 1/2. \end{cases}$
- $X = [0,1]$ and $C(x) = \begin{cases} \{1/2\} \text{ for } x < 1/2 \\ [1/3, 1/2[\text{ for } x = 1/2, \\ \{1/3\} \text{ for } x \geq 1/2. \end{cases}$

2. Let $f : S \times S \to \mathbb{R}$ be a continuous function where $S \subseteq \mathbb{R}^n$ is a compact and convex set. Assume that for all vectors $\mathbf{y} \in S$, the function $f(., \mathbf{y}) : S \to \mathbb{R}$ is concave, i.e. for all $\mathbf{x}, \mathbf{x}' \in S$ and all $\alpha \in [0,1]$,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}', \mathbf{y}) \geq \alpha f(\mathbf{x}, \mathbf{y}) + (1-\alpha) f(\mathbf{x}', \mathbf{y}).$$

For $\mathbf{y} \in S$, let $B(\mathbf{y})$ be the set of vectors $\mathbf{x}^* \in S$ that solve the following problem.

$$\max_{\mathbf{x} \in S} f(\mathbf{x}, \mathbf{y}).$$

Show that the correspondence B has a fixed point, i.e., there is a vector $\mathbf{x} \in S$ such that $\mathbf{x} \in B(\mathbf{x})$.

Solutions

Remark, regarding the proofs, more than one possible solution may be correct. Here I only provide one possibility.

Logic and proofs

1. The truth table is as follows.

p	q	r	$p \to q$	$(p \to q) \vee r$	$\neg p \vee q \vee r$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	1	1
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	1	1	1

   Using formal rules of deduction we have,

$$
\begin{aligned}
& (p \to q) \vee r, \\
\Longleftrightarrow\ & \neg\left(\neg\left[(p \to q) \vee r\right]\right) && (\text{ by double negation } w = \neg(\neg w)) \\
\Longleftrightarrow\ & \neg\left(\neg(p \to q) \wedge \neg r\right) && (\text{ from } \neg(w \vee s) = \neg w \wedge \neg s), \\
\Longleftrightarrow\ & \neg(p \wedge \neg q \wedge \neg r) && (\text{ from } \neg(p \to q) = p \wedge \neg q), \\
\Longleftrightarrow\ & \neg p \vee q \vee r && (\text{ from } \neg(w \wedge q) = \neg w \vee \neg q).
\end{aligned}
$$

2. Let's proof it by induction on the difference between n and 3. For $n = 4$, we have that,

$$\sum_{t=3}^{4} \delta^t = \delta^3 + \delta^4 = \frac{(1-\delta)(\delta^3 + \delta^4)}{1-\delta} = \frac{\delta^3 - \delta^5}{1-\delta}.$$

   Now assume that it holds for all n up to m and take the case where $n = m+1$ then

$$
\begin{aligned}
\sum_{t=3}^{m+1} \delta^t &= \sum_{t=3}^{m} \delta^m + \delta^{m+1}, \\
&= \frac{\delta^3 - \delta^{m+1}}{1-\delta} + \frac{(1-\delta)\delta^{m+1}}{1-\delta}, \\
&= \frac{\delta^3 - \delta^{m+2}}{1-\delta}.
\end{aligned}
$$

   as was to be shown.

Warming up

1. Let $a^* = \sup A$ and $b^* = \sup B$ which both exist as A and B are bounded. Then as a^* is an upper bound of A (for all $a \in A$, $a \leq a^*$) and b^* is an upper bound for B (for all $b \in B, b \leq b^*$), we have that for all $a \in A, b \in B$,

$$a + b \leq a^* + b^* = \sup A + \sup B.$$

So the number $\sup A + \sup B$ is an upperbound for C (given that a and b were chosen arbitrarily). This means that C is bounded from above, so it has a supremum, say $c^* = \sup C$. This supremum is also smaller or equal to any other upperbound of C, meaning that,

$$c^* \leq a^* + b^*.$$

Let us now show that also $a^* + b^* \leq c^*$. From the definition of a supremum and the set C, we have that for all $a \in A$ and $b \in B$,

$$a + b \leq c^*.$$

Then for all $a \in A$ and $b \in B$,

$$a \leq c^* - b.$$

This means that for all $b \in B$, the number $c^* - b$ is an upperbound for A. Given that a^* is the lowest upperbound, we have that for all $b \in B$,

$$a^* \leq c^* - b,$$

This also means that for all $b \in B$,

$$b \leq c^* - a^*.$$

As such, we see that the number $c^* - a^*$ is an upperbound for B. Given that b^* is the lowest upperbound, we have,

$$\begin{aligned} & b^* \leq c^* - a^*, \\ \iff & b^* + a^* \leq c^*, \end{aligned}$$

as was to be shown.

2. The function $f(x) = x^3 - x + 1$ is clearly continuous. So in order to apply the intermediate value theorem, and show the existence of a root, we only need to show that $f(-2)$ and $f(0)$ have opposite signs. Indeed,

$$\begin{aligned} & f(-2) = (-2)^3 - (-2) + 1 = -8 + 2 + 1 = -5, \\ & f(0) = 1. \end{aligned}$$

Fixed points

1. (short answer: the complete solution should contain the figures and the discussion of the other assumptions of Kakutani's fixed point theorem.) For the first correspondence, the domain is not closed, so not compact. For the second correspondence the domain is not bounded, so not compact. The third correspondence is not convex at $x = 1/2$. The last correspondence is not upper hemi-continuous at $x = 1/2$: taking a sequence converging to $x = 1/2$ from below, we see that for each x_t in this sequence $1/2 \in C(x_t)$, but $1/2 \notin C(1/2)$.

2. This is an application of Kakutani. The proof is very similar (and in fact a special case for the proof for the existence of a Nash equilibrium). First, We check that the conditions for Berge's maximization theorem are satisfied. First f is continuous. Also, the set S is compact so the feasibility correspondence $G(y) = S$ is upper and lower-hemicontinuous.

   From this, we conclude that $B : S \twoheadrightarrow S$ is upper hemi-continous. From convexity of S and concavity of f in x, it also follows that B is convex valued. As such, by Kakutani, B has a fixed point.