# Exam: MATHS-400,
# Mathematics and Economic Modelling

This is part of the exam of January 12, 2018

- The exam lasts 3 hours. There are 8 questions in total so you have on average 22 and a half minute per question.
- The total grade on this exam is on 18. The grades on this exam will be added tot the grading (on 2) for the assignments.
- If you use a result from the course in order to answer your question make sure that you argue why all assumptions to use the result are valid. For example, if you use Brouwer's fixed point theorem, you need to demonstrate that the domain $S$ is compact and convex and the function maps from $S$ to $S$ and is continuous.
- The exam is open book. This means that you may use the lecture and exercise notes. This does not allow the use of laptops, mobile phones or any other electronic devices.

## Logic and proofs (4pt)

1. (2pt) Negate the following sentences
   a. (0.5pt) If it is an apple, then it is not a banana.
   b. (0.5pt) All classroom have at least one chair that is broken
   c. (0.5pt) For all $x \in A$ we have $x \notin B$.
   d. (0.5pt) $\forall \varepsilon > 0, \exists \delta \geq 0, \exists T \in \mathbb{N}, \forall t \geq T : |x - x_t| < \delta \to |f(x) - f(x_t)| < \varepsilon$.
2. (2pt) Proof the following statements and indicate what type of proof that you use.
   a. (1pt) For any $x \in \mathbb{Z}$ if $(7x + 9)$ is even, then $x$ is odd.
   b. (1pt) There is no largest even integer.

## Sequences and limits (4pt)

1. (2pt) Let $A$ and $B$ be two closed subsets of $\mathbb{R}^n$.
   a. (1pt) Show that their union $A \cup B$ is closed.
   b. (1pt) Show that their intersection $A \cap B$ is also closed. (remark: notice that the empty set $\emptyset$ is also closed).

2. (2pt) Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. For any $n \in \mathbb{N}$, define the number,

$$y_n = \sup_{m\geq n}\{x_m\} = \sup\{x_n, x_{n+1}, \ldots\}.$$

In words, $y_n$ is the supremum over all numbers $x_m$ for $m$ greater or equal to $n$.

a. (1pt) Show that for all $n \in \mathbb{N}$, $y_n$ exists (i.e. is well-defined).

b. (1pt) Show that the sequence $(y_n)_{n\in\mathbb{N}}$ has a limit. (hint: can you show that $y_n$ is either non-increasing or non-decreasing).

c. (bonus point) Analogues, one can define the numbers

$$z_n = \inf_{m\geq n}\{x_m\},$$

and show that the sequence $(z_n)_{n\in\mathbb{N}}$ has a limit. Show that the limit of $(y_n)_{n\in\mathbb{N}}$ is greater or equal to the limit of $(z_n)_{n\in\mathbb{N}}$. Show that the two limits are the same if $(x_n)_{n\in\mathbb{N}}$ itself has a limit.

# Continuity and maximization (5pt)

1. (2pt) Let $f : \mathbb{R}^k_+ \to \mathbb{R}_{++}$ be a continuous function and define the correspondence $\Gamma : \mathbb{R}^k_+ \twoheadrightarrow \mathbb{R}_+$ where $\Gamma(\mathbf{x}) = [0, f(\mathbf{x})]$. Show that $\Gamma$ is continuous, i.e. upper and lower hemi-continuous.

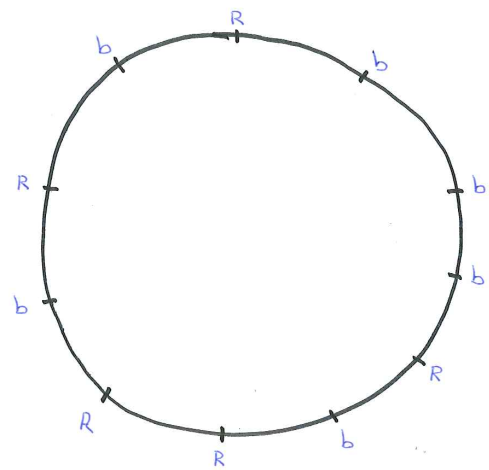
2. (3pt) Let $T = \mathbb{R}$, $X = \mathbb{R}$ and for all $\theta \in T$, $G(\theta) = \{x \in X : \theta - 3 \leq x \leq \theta + 3\}$. Define $f : X \to \mathbb{R}$ by $f(x) = |x|$. Consider the problem,

$$\max_x f(x) \text{ subject to } x \in G(\theta).$$

Solve this problem and draw the optimal value function, $v(\theta)$, and the optimal solution correspondence $\Gamma(\theta)$. Is the latter upper hemi-continous? lower hemi-continuous? convex valued?

# Fixed points (5pt)

1. (2.5pt) Consider a circle divided into a finite number of segments. Every endpoint of a segment is coloured in one of two colours, say red and blue, such that two segments with a common endpoint have the same colour for this endpoint. Assume that both colours are used in the colouring, so there is at least one blue and one red point. A segment is totally coloured if it's two endpoints have different colours. See the figure below for an illustration. Prove that the number of totally coloured segments is even and greater than or equal to two.



# Solutions

## Logic and proofs

1. Negation

   - Let $a$ be 'it is an apple' and $b$ 'it is a banana', then the sentence is

     $$a \to \neg b = \neg a \vee \neg b.$$

     Negating this gives

     $$\neg(\neg a \vee \neg b) = a \wedge b.$$

     negation is 'it is an apple and a banana'.

   - 'All classroom have at least one chair that is broken' can be written as

     $$\forall k \exists c \in k : B(c),$$

     where $k$ is a classroom, $c$ is a chair and $B(c)$ is true if the chair $c$ is broken. Negating this gives,

     $$\exists k \forall c \in k : \neg B(c),$$

     or equivalently 'there is a classroom where all chairs are not broken'.

   - The negation is,

     $$\exists x \in A, x \in B,$$

     or equivalently, $A \cap B \neq \emptyset$

   - $$\forall \varepsilon > 0, \exists \delta \geq 0, \exists T \in \mathbb{N}, \forall t \geq T, |x - x_t| < \delta \to |f(x) - f(x_t)| < \varepsilon.$$

     Negation gives,

     $$\exists \varepsilon > 0, \forall \delta \geq 0, \forall T \in \mathbb{N}, \exists t \geq T, |x - x_t| < \delta \wedge |f(x) - f(x_t)| \geq \varepsilon.$$

2. a. Let's use a proof by contrapositive. If $x$ is even, then $x = 2n$ for some number $n \in \mathbb{Z}$. Then $7x = 14n$ and $7x + 9 = 14n + 9 = 2(7n + 4) + 1$ which is an odd number. For b. the proof is by contradiction. Assume there is a largest even integer, say $n$. But then $n + 2$ is also even, and it is larger than the number $n$, which gives the desired contradiction.

## Sequences and Limits

1.

a. In order to show that $A \cup B$ is closed, we need to show that any convergent sequence in $A \cup B$ has a limit in $A \cup B$. Let $\mathbf{x}_n$ be a convergent sequence in $A \cup B$ with $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$. We need to show that $\mathbf{x} \in A \cup B$. Every element of the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is in either $A$ or $B$. As such, either an infinite number of elements of the sequence is in $A$, or an infinite number of element of the sequence is in $B$ (or both). This set of infinite elements forms a subsequence of $(\mathbf{x}_n)_{n \in \mathbb{N}}$ whose limit is also $\mathbf{x}$ (as any subsequence of a convergent sequence has the same limit). This subsequence is either in the closed set $A$ or in the closed set $B$ so its limit $\mathbf{x}$ is in either $A$ or $B$ which means that it is in $A \cup B$.

b. If $A \cap B$ is empty, the proof is done as the empty set is closed. Else, let $\mathbf{x}_n$ be a convergent sequence in $A \cap B$ with $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$. We need to show that $\mathbf{x} \in A \cap B$. Clearly the sequence is in both $A$ and $B$ so as both sets are closed, the limit $\mathbf{x}$ has to be in both $A$ and $B$, so it is in $A \cap B$.

2.

a. The sequence $(x_n)_{n \in \mathbb{N}}$ is bounded from above (and below), so $\{x_m : m \geq n\}$ is also bounded from above. Any set of real numbers which is bounded from above has a supremum, which shows that $y_n$ exists for all $n$.

b. Let us show that $y_n$ is non-increasing. We have that,

$$y_n = \sup_{m \geq n} x_n,$$
$$y_{n+1} = \sup_{m \geq n+1} x_n.$$

Now, $y_n \geq x_m$ for all $m \geq n$ so $y_n \geq x_m$ for all $m \geq n+1$. This means that $y_n$ is an upperbound for $\{x_m : m \geq n+1\}$. As $y_{n+1}$ is the lowest upperbound, we have that $y_n \geq y_{n+1}$. We can do this for any $n$, which shows that,

$$y_1 \geq y_2 \geq \ldots \geq y_n \geq \ldots$$

Also, for all $n$, $y_n \geq x_n \geq L$ for any lower bound $L$ of $\{x_n : n \in \mathbb{N}\}$. Given that $(x_n)_{n \in \mathbb{N}}$ is bounded from below, it follows that $(y_n)_{n \in \mathbb{N}}$ is also bounded from below. As such $(y_n)_{n \in \mathbb{N}}$ is a non-increasing sequence which is bounded from below, so it has a limit.

c. For all $n$, $y_n \geq x_n \geq z_n$ so $y_n \geq z_n$ for all $n$. Taking limits gives that $\lim_n y_n \geq \lim_n z_n$.

Now assume that $x_n \xrightarrow{n} x$. By definition of the supremum, for all $n$, there is an $m_n \geq n$ such that $y_n \leq x_{m_n} + \frac{1}{n}$. Also, by definition of the infimum, for all $n$ there is an $i_n \geq n$ such that $x_{i_n} - \frac{1}{n} \leq z_n$. Then,

$$x_{i_n} - \frac{1}{n} \leq z_n \leq y_n \leq x_{m_n} + \frac{1}{n}.$$

Taking the limit (along an increasing subsequence for $i_n$ and $m_n$) shows that

$$x \leq \lim_n z_n = \lim_n y_n \leq x,$$

so all limits are equal.

## Continuity and maximization

1. This question is very similar to exercise 2 of correspondences. (i) For uhc, notice first that as $f(\mathbf{x}) > 0$, $\Gamma(\mathbf{x})$ is non-empty. Next, $[0, f(\mathbf{x})]$ is clearly compact and closed as it is a closed interval. For the third property, let $\mathbf{x}_n \to \mathbf{x}$ and let $y_n \in \Gamma(\mathbf{x}_n)$ for all $n$. Let us first show that this sequence is bounded. Clearly, $0 \leq y_n$ for all $n$ which shows that $(y_n)_{n\in\mathbb{N}}$ is bounded from below. Also, by continuity of $f$, there is an $N$ such that for all $n \geq N$, $|f(\mathbf{x}_n) - f(\mathbf{x})| < 1$. Let us restrict the sequence to $(\mathbf{x})_{n\geq N}$. We see that for this sequence,

   $$0 \leq y_n \leq f(\mathbf{x}_n) \leq f(\mathbf{x}) + 1,$$

   so the sequence is bounded by $\max\{y_1, \ldots, y_{N-1}, f(\mathbf{x}) + 1\}$. Given that $(y_n)_{n\in\mathbb{N}}$ is bounded, it has a convergent subsequence, say $y_{n_i} \xrightarrow{i} y$. We will show show that $y \in [0, f(\mathbf{x})]$. Now, for all $n_i$ we have that,

   $$0 \leq y_{n_i} \leq f(\mathbf{x}_{n_i}).$$

   As $f$ is continuous, we have that, by taking limits,

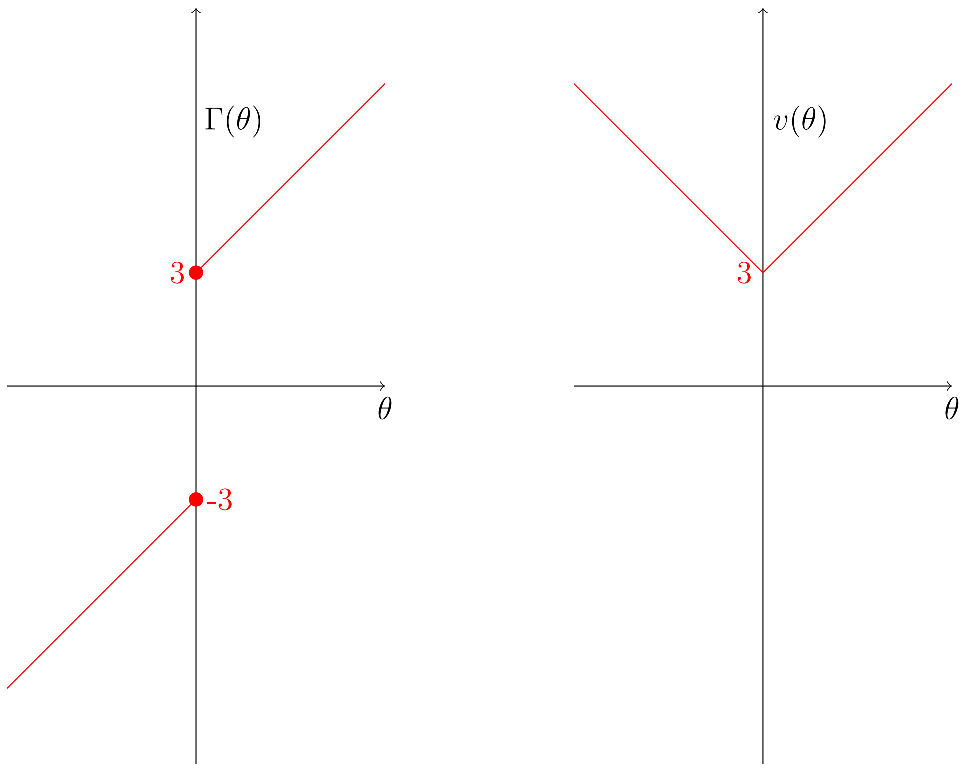
   $$0 \leq y \leq f(\mathbf{x}),$$

   which demonstrates the proof.

   For lhc, non-emptyness was established above. For the second property, let $y \in \Gamma(\mathbf{x})$. Then there is an $\alpha \leq 1$ such that $y = \alpha f(\mathbf{x})$. Let $\mathbf{x}_n \xrightarrow{n} \mathbf{x}$. Define $y_n = \alpha f(\mathbf{x}_n)$. Observe that $y_n \in \Gamma(\mathbf{x}_n)$ and that $y_n \xrightarrow{n} y$ as $f$ is continuous.

2. For $\theta < 0$ the optimal value of $x$ is given by $\theta - 3$. For $\theta > 0$ the optimal value for $x$ is given by $\theta + 3$. For $\theta = 0$ there are two optimal values for $x$ namely $x = \{-3, 3\}$. This gives,

   $$\Gamma(\theta) = \begin{cases} \{\theta - 3\} \text{ if } \theta < 0, \\ \{3, -3\} \text{ if } \theta = 0, \\ \{\theta + 3\} \text{ if } \theta > 0. \end{cases}$$




The optimal value function is given by $v(\theta) = |\theta| + 3$, which has a kink at 0 but is still continuous. The correspondence is uhc at all points $\theta$ which follows from Berge's maximum theorem (given that the $G(\theta)$ is continuous and that $f$ is continuous. It is not lhc at $\theta = 0$. To see this, consider the sequence $(1/n)_{n\in\mathbb{N}}$ that converges to 0. We have that $-3 \in \Gamma(0)$ but the only values in $\Gamma(1/n)$ are given by $1/n + 3$ which converge to 3 and not to $-3$. The correspondence $\Gamma$ is also not convex valued at 0 as 3 and $-3$ are in $\Gamma(0)$ but no convex combination of these two numbers is.

## Fixed points

1. Interpret each segment as a room with two walls and put a door in a wall with the colour blue. Put a doormat in front of every door. Then the number of doormats can be counted in two different ways. First every door has two doormats so, the number of doormats is equal to 2 times the number of doors. Next every $b - b$ room has two doormats, every $b - r$ has one doormat and every $r - r$ room has zero doormats. As such, the number of $b - r$ rooms is 2 times the number of doors minus 2 times the number of $b - b$ rooms, which shows that the number of $b - r$ rooms is even. Also given that the two different colours are used, it must be the case that there is at least one $b - r$ room.