Exam: MATHS-400,
Mathematics and Economic Modelling

This is part of the exam of January 13, 2017

- The exam lasts 3 hours. There are 8 questions in total so you have on average 22 and a half minute per question.
- The total grade on this exam is on 18. The grades on this exam will be added tot the grading (on 2) for the assignments.
- If you use a result from the course in order to answer your question make sure that you argue why all assumptions to use the result are valid. For example, if you use Brouwer's fixed point theorem, you need to demonstrate that the domain S is compact and convex and the function maps from S to S and is continuous.
- The exam is open book. This means that you may use the lecture and exercise notes. This does not allow the use of laptops, mobile phones or any other electronic devices.

Logic and proofs (4pt)

1. (2pt) Negate the following sentences
   a. (0.5pt) If it is an apple, then it is not a banana.
   b. (0.5pt) All classroom have at least one chair that is broken
   c. (0.5pt) For all $x \in A$ we have $x \notin B$.
   d. (0.5pt) $\forall \varepsilon > 0, \exists \delta \geq 0, \exists T \in \mathbb{N}, \forall t \geq T : |x - x_t| < \delta \rightarrow |f(x) - f(x_t)| < \varepsilon$.
2. (2pt) Proof the following statements and indicate what type of proof that you use.
   a. (1pt) For any $x \in \mathbb{Z}$ if $(7x + 9)$ is even, then x is odd.
   b. (1pt) There is no largest even integer.

Warming up (4pt)

1. (3pt)

   a. (1pt) Let A, B be two closed subsets of $\mathbb{R}^n$. Show that their union, $A \cup B$ is also closed.

   b. (1pt) Let $\{A_1, A_2, \ldots, A_m\}$ be a finite collection of closed subsets of $\mathbb{R}^n$. Using the statement in (a.), show that $\cup_{i=1}^m A_i$ is also closed. What kind of proof do you use?

   c. (1pt) Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be an infinite collection of closed sets. Can you give a counterexample to show that $\cup_{i \in \mathbb{N}} A_i$ is not necessarily closed.

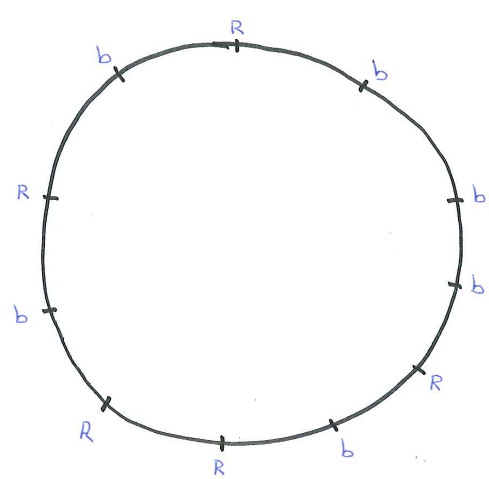
2. (1pt) Let C be a subset of $\mathbb{R}^n$ and let $\{\mathbf{x}_t\}_{t \in \mathbb{N}}$ be a sequence in C. Assume that every **convergent** subsequence of $\{\mathbf{x}_t\}_{t \in \mathbb{N}}$ has the same limit $\mathbf{x}$.

   a. (1pt) Show by counterexample that even if $\{\mathbf{x}_t\}$ has a convergent subsequence, it is not necessarily the case that the sequence $\{\mathbf{x}_t\}_{t \in T}$ converges to $\mathbf{x}$.

   b. (Bonus point) Assume now that C is compact. Show that in this case $\{\mathbf{x}_t\}_{t \in \mathbb{N}}$ is convergent and that $\mathbf{x}_t \to \mathbf{x}$.

Fixed points

1. (2.5pt) Consider a circle divided into a finite number of segments. Every endpoint of a segment is coloured in one of two colours, say red and blue, such that two segments with a common endpoint have the same colour for this endpoint. Assume that both colours are used in the colouring, so there is at least one blue and one red endpoint. A segment is totally coloured if it's two endpoints have different colours. See the figure below for an illustration. Prove that the number of totally coloured segments is even and greater than or equal to two.



Solutions

Logic and proofs

1. Negation

   - Let a be 'it is an apple' and b 'it is a banana', then the sentence is

     $$a \to \neg b = \neg a \vee \neg b.$$

     Negating this gives

     $$\neg(\neg a \vee \neg b) = a \wedge b.$$

     negation is 'it is an apple and a banana'.
   - 'All classroom have at least one chair that is broken' can be written as

     $$\forall k \exists c \in k : B(c),$$

     where k is a classroom, c is a chair and $B(c)$ is true if the chair c is broken. Negating this gives,

     $$\exists k \forall c \in k : \neg B(c),$$

     or equivalently 'there is a classroom where all chairs are not broken'.
   - The negation is,

     $$\exists x \in A, x \in B,$$

     or equivalently, $A \cap B \neq \emptyset$
   - $$\forall \varepsilon > 0, \exists \delta \geq 0, \exists T \in \mathbb{N}, \forall t \geq T, |x - x_t| < \delta \to |f(x) - f(x_t)| < \varepsilon.$$

     Negation gives,

     $$\exists \varepsilon > 0, \forall \delta \geq 0, \forall T \in \mathbb{N}, \exists t \geq T, |x - x_t| < \delta \wedge |f(x) - f(x_t)| \geq \varepsilon.$$

2. Let's use a proof by contrapositive. If x is even, then $x = 2n$ for some number $n \in \mathbb{Z}$. Then $7x = 14n$ and $7x + 9 = 14n + 9 = 2(7n + 4) + 1$ which is an odd number.

Warming up

1. Let A, B be closed. In order to show that $A \cup B$ is closed, we can show that $\mathbb{R}^n \setminus (A \cup B)$ is open. This is the set $(\mathbb{R}^n \setminus A) \cap (\mathbb{R}^n \setminus B)$, which is the intersection of two open sets.

   For ease of notation let $\bar{A} = \mathbb{R}^n \setminus A$ and $\bar{B} = \mathbb{R}^n \setminus B$. Let $x \in \bar{A} \cap \bar{B}$. Given that $\bar{A}$ and $\bar{B}$ are open, there are numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that,

   $$B_{\varepsilon_1}(x) \subseteq \bar{A},$$
   $$B_{\varepsilon_2}(x) \subseteq \bar{B},$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. We have that $B_\varepsilon(x) \subseteq \bar{A} \cap \bar{B}$ showing that $\bar{A} \cap \bar{B}$ is open.

Let $A_i, i = 1, < ldots, m$ be a finite collection of closed sets. If $m = 1$ then evidently $\cup_i A_i = A_1$ is closed. Let us show that $\cup_{i=1}^m A_i$ is closed by induction. Assume that the result holds up to m and consider the case $m + 1$, then $\cup_{i=1}^{m+1} A_i = \cup_{i=1}^m A_i \cup A_{m+1} = B \cup A_{m+1}$. The set B is closed by the induction hypothesis, so this gives the union of two closed sets, which we know by the previous result is also closed.

For the counterexample consider the sets $A_i = [1, 1 - 1/i]$, then $\bigcup_{i \in \mathbb{N}} = A_i = [0, 1[$ which is not closed.

2. Consider for example the sequence $\{x, 1, x, 2, x, 3, x, 4, x, 5, x, 6, \ldots, \}$. Then any convergent subsequence has limit x but the entire sequence is unbounded from above, so it has no limit.

   For the bonus question, consider a proof by contradiction. If $\mathbf{x}_t$ does not converge to $\mathbf{x}$, then there exists a $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists a $t \geq N$ such that $\|\mathbf{x} - \mathbf{x}_t\| \geq \varepsilon$. This creates a subsequence $\{\mathbf{x}_{t_i}\}_{i \in \mathbb{N}}$. Such that for each t_i, $\|\mathbf{x} - \mathbf{x}_{t_i}\| \geq \varepsilon$. This subsequence lies in a compact set, so it also has a subsequence, which converges by Bolzano-Weierstrass. This subsequence is also a subsequence of $\{\mathbf{x}_t\}$ so, by assumption it should converge to $\mathbf{x}$. However this cannot be the case as for each term in this sub-subsequence $\|\mathbf{x} - \mathbf{x}_{t_i}\| \geq \varepsilon$. This gives the contradiction.

Fixed points

1. Interpret each segment as a room with two walls and put a door in a wall with the colour blue. Put a doormat in front of every door. Then the number of doormats can be counted in two different ways. First every door has two doormats so, the number of doormats is equal to 2 times the number of doors. Next every $b - b$ room has two doormats, every $b - r$ has one doormat and every $r - r$ room has zero doormats. As such, the number of $b - r$ rooms is 2 times the number of doors minus 2 times the number of $b - b$ rooms, which shows that the number of $b - r$ rooms is even. Also given that the two different colours are used, it must be the case that there is at least one $b - r$ room.