MATH-S400

Assignment 4: Fixed points

1. Consider a Cournot oligopoly where the inverse demand function is given by $P(q_1 + q_2) = a - b(q_1 + q_2)$ with $a, b > 0$. Let both firms have equal marginal costs given by $c > 0$ (assume that $b < a - c$). The profit functions of the firms is given by:

$$\pi_1(q_1, q_2) = P(q_1 + q_2)q_1 - cq_1,$$
$$\pi_2(q_1, q_2) = P(q_1 + q_2)q_2 - cq_2,$$

   Show that there is a Nash equilibrium.

2. Let $z : \Delta^{J-1} \to \mathbb{R}^J$ be the excess demand function for an exchange economy with J goods. Remember that for all price vectors $\mathbf{p} \in \Delta^{J-1}$, $\mathbf{p} \cdot z(\mathbf{p}) = 0$ by Walras' law.

   We say that the excess demand function satisfies the Weak Axiom of Revealed Preference if for any two price vectors $\mathbf{p}, \mathbf{p}' \in \Delta^{J-1}$:

$$(z(\mathbf{p}) \neq z(\mathbf{p}') \text{ and } \mathbf{p} \cdot z(\mathbf{p}') \leq 0) \text{ implies } \mathbf{p}' \cdot z(\mathbf{p}) > 0.$$

   A price vector $\mathbf{p}^*$ is called an equilibrium price vector if $z(\mathbf{p}^*) = \mathbf{0}$ (i.e. all markets are in equilibrium). Show that if the Weak Axiom of Revealed Preference is satisfied, then the set of all equilibrium price vectors is a convex set.

3. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^\ell$ be two sets such that A is convex and compact. Let $f : A \to B$ be a continuous function and let $g : B \to A$ be a continuous function. Show that there are elements $\mathbf{x} \in A$ and $\mathbf{y} \in B$ that map to each other in the sense that $f(\mathbf{x}) = \mathbf{y}$ and $g(\mathbf{y}) = \mathbf{x}$.

4. Let $f(x, y) : S \times S \to \mathbb{R}$ be a continuous function which is quasi-convex in x and quasi-concave in y and where S is compact and convex. Show that there is a point $(x^*, y^*) \in S \times S$ such that for all $(x, y) \in S \times S$:

$$f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*).$$

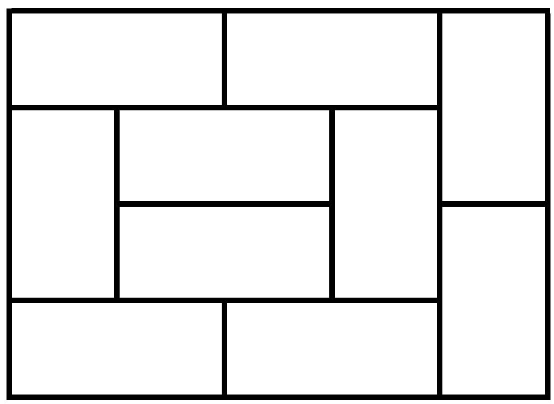
   In other words, x^* is a solution to the problem,

$$\min_{x \in S} f(x, y^*),$$

and y^* is a solution to the problem

$$\max_{y \in S} f(x^*, y).$$

5. Consider a rectangle that is filled with smaller rectangles. Assume that for each of these smaller rectangles has either width 1 and height 2 or width 2 and height 1 as in the picture below.



Show that it is impossible that such big rectangle has both an odd width and odd height. (Hint: try a proof by contradiction. Divide the small rectangles into triangles then colour each vertex with coordinates (x, y) in one of three colors according to (i) x being even, (ii) x being odd and y being even and (iii) both x and y being odd. What does this tell us about the colours of the vertices on the outside of the rectangle? Use this to show that there is a small triangle with three different colours? Is this possible?)