MATH-S400
Assignment 3

1. Let $D, R \subseteq \mathbb{R}^k$ be two sets and let $F : D \twoheadrightarrow R$ and $G : D \twoheadrightarrow R$ be two correspondences that are both upper and lower hemi-continuous. Consider the correspondence $K : D \twoheadrightarrow R$ where for all $x \in D$:

   $$K(\mathbf{x}) = F(\mathbf{x}) \cup G(\mathbf{x}).$$

   Show that the correspondence $K : D \twoheadrightarrow R$ is also both upper and lower hemi-continuous.

2. Let $f : \mathbb{R} \to [a, b]$ be a continuous function such that:

   $$\forall x \in \mathbb{R} : f(x) = 0 \leftrightarrow x^2 = 1.$$

   Show that if $f(2) > 0$ then $f(4) > 0$.

3. Let $\theta \in \mathbb{R}$ and consider the problem:

   $$\max_x \theta x \text{ s.t. } x \in \{-1, 0, 1\}.$$

   Are the assumptions of Berge's maximization theorem satisfied? Solve this problem and graph the optimal value function and the solution correspondence. Is this correspondence upper hemi-continuous, lower hemi-continuous? why/why not?

4. On monday, Al Pine starts climbing the Mont Blanc. He starts at 8am in the morning and stops at 8pm in the evening. He stays the night on the mountain and decides to go back down the mountain the next day (as he forgot to bring a scarf). He starts descending at 7am and arrives back at the foot of the mountain at 3pm.

   Show that there is a time t and an altitude a such that on both Monday and Tuesday at that particular time (t), Al was at exactly the same altitude (a).

5. Let $f : X \times Y \to \mathbb{R}$ be a continuous and concave function where $X \subseteq \mathbb{R}^k$ is compact and convex set and $Y \subseteq \mathbb{R}^k$ is convex. Define the optimal value function:

   $$v(\mathbf{y}) = \max_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \text{ s.t. } \mathbf{x} \in X.$$

   Show that $v : Y \to \mathbb{R}$ is a concave function.