MATH-S400
Assignment 2

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $g : [a, b] \to \mathbb{R}$ be a continuous function such that for all $x \in [a, b]$, $g(x) > 0$. Show that there exists a $c \in [a, b]$ such that:

$$f(c) \int_a^b g(x)dx = \int_a^b f(x)g(x)dx.$$

2. Show that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ then for all $k \in \mathbb{N}$, $\left((x_n)^k\right)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathbb{R}$, where $(x_n)^k = \underbrace{x_n\ x_n \ldots x_n}_{k \text{ times}}$.

3. Let $\mathbf{p}$ be a vector in $\mathbb{R}^k$ and consider the half-space:

$$A = \{\mathbf{x} \in \mathbb{R}^k : \mathbf{p} \cdot \mathbf{x} < 2\}$$

Show that A is an open set.

4. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $h_n \overset{n}{\to} 0$. The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at the point $x \in \mathbb{R}$ is defined as the limit of the sequence:

$$y_n = \left(\frac{f(x + h_n) - f(x)}{h_n}\right).$$

Use this definition to show that the derivative of the function $f(x) = x^2$ is equal to $2x$.

5. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions such that for all $\mathbf{x} \in A$, $f(\mathbf{x}) \geq 0$ and $g(\mathbf{x}) \geq 0$. Show that:

$$\sup\{f(\mathbf{x})\, g(\mathbf{x}) : \mathbf{x} \in A\} \leq (\sup\{f(\mathbf{x}) : \mathbf{x} \in A\})\, (\sup\{g(\mathbf{x}) : \mathbf{x} \in A\}).$$

In words: the supremum of the product is less than or equal to the product of the suprema.