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DYNAMIC OPTIMIZATION

Contents

An example

Dynamic optimization problems deal with the following class of problems,

$$\begin{aligned} &\max_{\{a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, a_t), \\ \text{subject to } & x_t \in X, a_t \in A, \\ & a_t \in \Gamma(x_t), \\ & x_{t+1} = r(x_t, a_t), \\ & x_0 \text{ given.} \end{aligned}$$

Such problems contain the following ingredients.

1. A set of states, denoted by X and a set of actions, given by A. A state is denoted by $x \in X$ and an action is denoted by $a \in A$.
2. A correspondence $\Gamma : X \to A$ that determines for each state $x \in X$, which actions $a \in \Gamma(x)$ can be taken by the decision maker. given that the state is x.[1]
3. An instantaneous payoff function $F(x, a)$ that determines the immediate benefit of taking an action $a \in A$ when the state is $x \in X$.
4. A transition function $r : X \times A \to X$ where $r(x, a)$ gives the state $y = r(x, a) \in X$ in the next period given that the current state is $x \in X$ and the current action is $a \in A$.
5. A discount rate $\beta \in (0, 1)$ that determines the trade-off between future and current payoffs.

[1] Think $\Gamma(x)$ as a budget constraint where x is the vector of prices and income and a is the consumption bundle.

The problem is to determine the optimal (infinite) sequence of actions $a_0, a_1, \ldots$ that should be taken in order to optimize the infinite horizon payoff function

$$\max_{\{a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, a_t),$$

Although this problem may look like a standard optimization problem, there is one key difference. Namely, the optimization problem requires us to find an infinite number of values $\{a_t\}_{t=0}^{\infty}$ rather than a finite number of values. As such, it is not certain that the usual approach to solve standard optimization problems can also be used to solve this problem.[2]

[2] By usual, we mean the act of setting up the Lagrangean and take the corresponding Kuhn-Tucker first order conditions.

Before we attack the problem in full force, let us start by considering an example. We will choose the Ramsey-Cass-Koopmans model which extended the famous Solow model by permitting elastic savings rates.[3] The Ramsey-Cass-Koopman model is a representative consumer model with endogenous capital formation. In this model, capital is the only input in the production process. The output for a given amount of capital k is determined by a production function

[3] Ramsey, Frank P. (1928), "A Mathematical Theory of Saving," Economic Journal. 38: 543-559.
Cass, David, (1965), "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies. 32: 233-240.
Koopmans, T. C., (1965), "On the Concept of Optimal Economic Growth," The Economic Approach to Development Planning. Chicago: Rand McNally. pp. 225-287.

$$f(k) = Ak^{\alpha}.$$

Where $\alpha \in (0,1)$ is the output elasticity of capital. There is a representative household that chooses a sequence of consumption levels $\{c_t\}_{t=0}^{\infty}$. The period t payoff of choosing c_t gives an instantaneous payoff of

$$u(c_t) = \ln(c_t).$$

The problem faced by the representative household is to choose a sequence $\{c_t\}_{t=0}^{\infty}$ of consumption levels and a corresponding sequence of capital holdings $\{k_t\}_{t=0}^{\infty}$ to maximize discounted lifetime utility,

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t).$$

where $\beta \in (0,1)$ is an exogenous discount rate. Capital generation is determined by the following law of motion,

$$k_{t+1} = Ak_t^{\alpha} - c_t.$$

Here k_{t+1} is the stock of capital in period $t+1$. It is equal to the total amount of output, $f(k_t)$, minus the part of output that is used for immediate consumption, c_t. There is a clear trade off. Consumption increases the payoff but decreases future consumption by lowering the next period's amount of capital. The final piece of information to set up the model is a fixed initial level of capital k_0. Combining all pieces, we obtain the following problem,

$$\begin{aligned}
\max_{\{c_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \ln(c_t), \\
\text{s.t. } & k_{t+1} = Ak_t^{\alpha} - c_t, \\
& k_t, c_t \geq 0, \\
& k_0 \text{ given.}
\end{aligned}$$

Translating this into the dynamic optimization framework from the beginning of the chapter, we obtain the following ingredients.

1. The state space X is given by the feasible amounts of capital, $(= \mathbb{R}_+)$. A state is given by a stock of capital $k \in X$. The action space, A, is the possible set of consumption levels $(= \mathbb{R}_+)$. An action is an amount of consumption $c \in A$.

2. The correspondence $\Gamma(k)$ determines the possible consumption levels when the level of capital is equal to k. It is determined by,

$$\Gamma(k) = \{c \in \mathbb{R}_+ : c_t \leq Ak^\alpha\}.$$

3. The instantaneous payoff function is given by, $F(k, c) = \ln(c)$. In this setting, it is independent of k (for given c).

4. The transition function $r(k, c)$ that determines the next periods amount of capital is given by $r(k, c) = Ak^\alpha - c$.

5. The discount rate is given by β.

It is instructive to first solve this problem when the time horizon is finite instead of infinite. Let T be the final period. If $T = 0$, we obtain a static optimization problem whose solution depends on the initial capital stock k_0.

$$v_0(k_0) = \max_{c_0} \ln(c_0) \text{ s.t. } k_1 = Ak_0^\alpha - c_0; k_1, c_0 \geq 0.$$

Given that $k_1 \geq 0$ and $\ln(.)$ is strictly increasing, the optimal solution is to set $k_1 = 0$ and $c_0 = Ak_0^\alpha$.[4] The function $v_0(k_0)$ is called the value function. It only depends only on the initial capital stock as all future capital stocks are determined by the optimal choice of the consumption levels. Substituting $k_1 = 0$ and $c_1 = Ak_0^\alpha$ into the problem gives,

[4] As positive amounts of k_1 generate no additional utility, it is optimal to leave no money on the table after the final period.

$$v_0(k_0) = \ln(Ak_0^\alpha) = \ln(A) + \alpha \ln(k_0).$$

Now, let look at the problem when the final time period $T = 1$. In this case, we need to choose two consumption levels c_0 and c_1.

$$\begin{aligned} v_1(k_0) = &\max_{c_0, c_1} \{\ln(c_0) + \beta \ln(c_1)\}, \\ \text{s.t. } & k_1 = Ak_0^\alpha - c_0, \\ & k_2 = Ak_1^\alpha - c_1, \\ & c_0, c_1, k_1, k_2 \geq 0. \end{aligned}$$

Given that $k_2 \geq 0$, one clearly sees that $k_2 = 0$ should hold at the optimum.[5] Given this, we can substitute the constraints $c_1 = Ak_1^\alpha$ and

[5] Again, there should be no money left on the table.

$c_0 = Ak_0^\alpha - k_1$ into the objective function.

$$v_1(k_0) = \max_{k_1}\{\ln(Ak_0^\alpha - k_1) + \beta \ln(Ak_1^\alpha)\}.$$

The first order condition gives,

$$\begin{aligned}
&- \frac{1}{Ak_0^\alpha - k_1} + \beta\alpha \frac{Ak_1^{\alpha-1}}{Ak_1^\alpha} = 0,\\
\rightarrow &- \frac{1}{Ak_0^\alpha - k_1} + \beta\alpha \frac{1}{k_1} = 0,\\
\rightarrow &k_1 = \frac{\alpha\beta}{1+\alpha\beta} Ak_0^\alpha.
\end{aligned}$$

The last line gives the optimal solution for k_1. Plugging this solution back into the objective function gives the value of $v_1(k_0)$.

Observe that here $Ak_0^\alpha - k_1 = \frac{1}{1+\alpha\beta}Ak_0^\alpha = c_0 \geq 0$, so the constraints $c_0, c_1, k_1 \geq 0$ are satisfied. Additionally, it is easily verified that the objective function is concave in k_1, so the solution characterized by the first order conditions is a global maximum.

$$\begin{aligned}
v_1(k_0) &= \ln\left(Ak_0^\alpha - \frac{\alpha\beta}{1+\alpha\beta}Ak_0^\alpha\right) + \beta \ln\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}Ak_0^\alpha\right)^\alpha\right),\\
&= \ln\left(\frac{A}{1+\alpha\beta}\right) + \alpha \ln(k_0) + \beta \ln\left(\frac{A^{1+\alpha}(\alpha\beta)^\alpha}{(1+\alpha\beta)^\alpha}\right) + \alpha^2\beta \ln(k_0),\\
&= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln\left(\frac{A^{1+\alpha}(\alpha\beta)^\alpha}{(1+\alpha\beta)^\alpha}\right) + \alpha(1+\alpha\beta)\ln(k_0),
\end{aligned}$$

So far so good. extending the final period once more, we set $T = 2$. Then we can write the problem as,[6]

$$v_2(k_0) = \max_{k_1,k_2}\{\ln(Ak_0^\alpha - k_1) + \beta \ln(Ak_1^\alpha - k_2) + \beta^2 \ln(Ak_2^\alpha)\}.$$

[6] In this case, we can set $k_2 = 0$ and substitute the constraints into the objective function.

The two first order conditions are,

$$\begin{aligned}
\frac{1}{Ak_0^\alpha - k_1} &= \frac{\alpha\beta Ak_1^{\alpha-1}}{Ak_1^\alpha - k_2},\\
\frac{1}{Ak_1^\alpha - k_2} &= \frac{\alpha\beta^2 Ak_2^{\alpha-1}}{Ak_2^\alpha}.
\end{aligned}$$

The solution is,[7]

$$\begin{aligned}
k_1 &= \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}Ak_0^\alpha,\\
k_2 &= \frac{\alpha\beta}{1+\alpha\beta}Ak_1^\alpha.
\end{aligned}$$

[7] It is readily verified that this implies $c_0, c_1, c_2, k_1, k_2 \geq 0$. Also the objective function is concave in (k_1, k_2) so the first order conditions are sufficient for a global maximum.

Substituting these solutions into the objective function gives the value function $v_2(k_0)$. This expression is big mess.[8] We can iterate this procedure, and solve the problem for $T = 3, 4, 5, \ldots$. Doing this, it can be shown that the solution converges for $T \rightarrow \infty$ to the values,

[8] Try it.

$$\begin{aligned}
&v_\infty(k_0) = a + b\ln(k_0), \text{ with,}\\
&a = \frac{1}{1-\beta}\left[\ln(A(1-\alpha\beta)) + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right],\\
&b = \frac{\alpha}{1-\alpha\beta}.
\end{aligned}$$

This motivate the following procedure to solve the infinite horizon maximization problem: repeatedly solve the dynamic optimization problem for T finite, i.e. $T = 0, 1, 2, 3, \ldots$, and look whether the solution converges when considering $T \to \infty$.

There are several problems with this approach. First of all, it is is not sure whether we will always get a clean functional form for $v_t(k_0)$. In our special setting where $f(k) = Ak^\alpha$ and $u(c) = \ln(c)$, we did have a closed form expression, but this is not the case in general. If we don't have a closed form solution for $v_t(k_0)$ is not clear how we should proceed. Second, even if we obtain a closed form solution, the method is rather cumbersome. We need to solve the optimization problem for various time periods in order to see some convergence going on. Third, even assuming that we are able to solve the problem for several finite time periods, it is not certain that these solutions convergence to some limiting solution. Let alone that we are able to proof such convergence. Fourth, even if this convergence happens, nothing guarantees us that the limit of the finite horizon optimization problem also provides a solution for the infinite horizon problem. Finally, we have no idea that this limit solution is the unique solution.

Given the large number of unresolved issues, it might be a good idea to have a fresh look at the initial problem.

$$v(k_0) = \max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^\infty \ln(c_t) \text{ s.t. } c_t + k_{t+1} \leq f(k_t),$$

Let us rewrite this problem by splitting it up into two sub-problems. The problem of deciding on the initial amount of consumption c_0 and the decision of choosing all future consumption levels $(c_1, c_2, \ldots)$ given our initial choice of c_0.

$$\begin{aligned}
v(k_0) &= \max_{c_t \leq Ak_t^\alpha} \sum_{t=0}^\infty \ln(c_t) \text{ s.t. } k_{t+1} = Ak_t^\alpha - c_t, \\
&= \max_{c_0 \leq Ak_0^\alpha} \left\{ \ln(c_0) + \beta \max_{c_t \leq Ak_t^\alpha} \sum_{t=1}^\infty \beta^{t-1} \ln(c_t) \right\} \text{ s.t. } k_{t+1} = Ak_t^\alpha - c_t, \\
&= \max_{c_0 \leq Ak_0^\alpha} \{\ln(c_0) + \beta v(k_1)\} \text{ s.t. } k_1 = Ak_0^\alpha - c_0
\end{aligned}$$

This shows that we can reformulate the infinite horizon problem as a recursive problem. The optimal value $v(k_0)$ for an initial capital stock k_0 is determined by choosing c_0 to maximize current payoff $\ln(c_0)$ and the value of the future payoff which is conveniently written down as $\beta v(k_1) = \beta v(Ak_0^\alpha - c_0)$. The functional equation

$$v(k) = \max_{c \leq Ak^\alpha} \{\ln(c) + \beta v(Ak^\alpha - c)\}.$$

is called the **Bellman equation** of the dynamic optimization problem.[9] If we could somehow find out the value of the function $v(.)$, we could simply insert it into the right hand side, maximize this right hand side with respect to c and find out the optimal value for c for any initial level of capital k.

[9] A functional equation is an equation of where the unknown is an entire function instead of a single variable.

One way to find out $v(.)$ is to make an educated guess. Before, we say that the limiting value of the value function of the finite horizon problem was of the form $v(k) = a + b\ln(k)$. Substituting this into the Bellman equation gives,

$$a + b\ln(k) = \max_c\{\ln(c) + a\beta + b\beta\ln(Ak^\alpha - c)\},$$

The maximization problem on the right hand side gives the following first order conditions,[10]

[10] We see that the right hand side is concave in c and $c \geq 0$ so the first order conditions give a global maximum.

$$\begin{aligned}\frac{1}{c} - \frac{\beta b}{Ak^\alpha - c} &= 0,\\ \rightarrow c &= \frac{Ak^\alpha}{1+\beta b},\\ \rightarrow Ak^\alpha - c &= \frac{\beta b}{1+\beta b}Ak^\alpha.\end{aligned}$$

Plugging this into the Bellman equation gives,

$$a + b\ln(k) = \ln\left(\frac{Ak^\alpha}{1+\beta b}\right) + a\beta + b\beta\ln\left(\frac{\beta b}{1+\beta b}Ak^\alpha\right),$$

Matching up the coefficients on $\ln(k)$ gives,

$$\begin{aligned}b &= \alpha(1+\beta b),\\ \rightarrow b &= \frac{\alpha}{1-\alpha\beta}.\end{aligned}$$

Matching up the constants gives,

$$a = \ln\left(\frac{A}{1+\beta b}\right) + \beta a + \beta b\ln\left(\frac{\beta b A}{1+\beta b}\right).$$

Substituting for b and solving for a finally gives,

$$a = \frac{1}{1-\beta}\left[\ln(A(1-\alpha\beta)) + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right],$$

This gives the same solution as before. In this case, we do have a closed form solution for the value function and for every initial capital stock k we know the optimal consumption level $c = \frac{Ak^\alpha}{1+\beta b}$. As such, the optimal solution can be found iteratively,

$$
\begin{aligned}
&c_0 = \frac{Ak_0^\alpha}{1+\beta b}, k_1 = Ak_0^\alpha - c_0,\\
&c_1 = \frac{Ak_1^\alpha}{1+\beta b}, k_2 = Ak_1^\alpha - c_0,\\
&\ldots,\\
&c_t = \frac{Ak_t^\alpha}{1+\beta_b}, k_{t+1} = Ak_t^\alpha - c_t,\\
&\ldots
\end{aligned}
$$

The tricky part of this approach is, obviously, that we have to guess the functional form of the value function $v(.)$ and there are only a few very specific instances where we can make a good guess about this functional form.

WHAT THEN SHOULD we do if we don't know the form of the value function. Let's go back to the Bellman equation.

$$v(k) = \max_{c \le f(k)} \{u(c) + \beta v(f(k) - c)\}.$$

Can we still somehow use this equation to solve our problem. The answer is yes and the key to the solution lies in the 'recursiveness' of the equation.

Assume that we start with an "arbitrary" guess for the function $v(.)$, say $v_0(.)$. We know that v_0 does not satisfy the Bellman equation, but let us substitute it into the right hand side anyway. Doing this gives us on the left hand side a new function, say $v_1(.)$.[11]

[11] Observe that we start with a function $v_0(.)$ and get an entire new function $v_1(.)$ out of this by varying the level of k on the left and right hand side.

$$v_1(k) = \max_{c \le f(k)} \{u(c) + \beta v_0(f(k) - c)\}.$$

Now, we can do the same thing with $v_1(.)$: plug it into the right hand side of the Bellman equation and look at the values that it generates on the left hand, giving us a new function $v_2(.)$.

$$v_2(k) = \max_{c \le f(k)} \{u(c) + \beta v_1(f(k) - c)\}.$$

We can continue this process indefinitely, and generate functions $v_1(.), v_2(.), v_3(.), \ldots, v_n(.), \ldots$ What happens if we allow $n \to \infty$. We would hope that finally the function $v_n(.)$ converges to some limiting function v_∞ that satisfies our Bellman equation,

$$v_\infty(k) = \max_{c \le f(k)} \{u(c) + \beta v_\infty(f(k) - c)\}.$$

This is the function we were looking for all along. Of course, currently, we don't know whether this iteration will converge to something useful or even that different starting functions for $v_0(.)$ will converge to the same limiting function $v_\infty(.)$. Studying the conditions for which this iteration does converge is a main part of the theory of dynamic programming.

Mathematical Preliminaries

In this chapter, we will introduce the necessary mathematical tools and results for the following chapters. We will need to have a look at the concepts of vector spaces, metric spaces and normed vector spaces. A special subclass of these spaces have the property that every Cauchy sequence has a limit, called Banach spaces.

Banach spaces will provide the necessary structure for our state space. We will define contraction mappings on these spaces and show that these have a unique fixed point. Additionally, we will present a useful result called Blackwell's theorem that gives an easy to verify set of conditions for a mapping to be a contraction mapping.

In a second part of the chapter, we will have a look at the theorem of the maximum. This celebrated result in economics gives us conditions for which the result of a maximization exists and satisfies some convenient continuity conditions.

Banach spaces

Before we can introduce the concept of a Banach space, we first need to define vector spaces.

Definition 1 (vector space). *A **real vector space** X is a set of elements together with two operations, addition and scalar multiplication.*[12] *For any two vectors $x, y \in X$, addition gives a vector $x + y \in X$ and for any vector $x \in X$ and a real number $\alpha \in \mathbb{R}$, scalar multiplication gives $\alpha x \in X$. We have the following 8 conditions,*

[12] The adjective real simply indicates that scalar multiplication is defined taking the reals, not elements of the complex plane or some other set.

1. $x + y = y + x;$
2. $(x + y) + z = x + (y + z)$
3. $\alpha(x + y) = \alpha x + \alpha y,$
4. $(\alpha + \beta)x = \alpha x + \beta x,$
5. $(\alpha\beta)x = \alpha(\beta x).$
6. $1x = x.$

Additionally, there is a zero element $\theta \in X$ such that,

7. $x + \theta = x$,

8. *for every $x \in X$ there is a $-x$ such that $x + (-x) = \theta$*

A first well known example of a vector space is the set of n-dimensional real vectors $\mathbb{R}^n$. However, the concept of a vector space is much broader than vectors of numbers. We will mainly work with vector spaces that have real valued functions as elements. Consider two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ defined on some common domain X. Then we can define their sum $f + g$ as the function,

$$(f + g)(x) = f(x) + g(x),$$

and the scalar product, αf ($\alpha \in \mathbb{R}$),

$$(\alpha f)(x) = \alpha f(x).$$

It is clear that these operations satisfy all eight conditions of a vector space.[13] As such, the set $F(X)$ of real valued functions on a common domain forms a vector space. We can actually go further. If X is a topological space and if f and g are continuous functions then $f + g$ and αf are also continuous functions, so the set of all continuous real valued functions with domain X is also a vector space. Let us call this space $C(X)$.

[13] Here we define the null-vector θ to be the function $\theta(x) = 0$ for all $x \in X$.

The following are real vector spaces:

- The finite Euclidean space $\mathbb{R}^n$
- The set $X = \{x \in \mathbb{R}^2 : x = \alpha z\}$, where $z \in \mathbb{R}^2$
- The set of all continuous functions on $[a, b]$,

The following are not vector spaces

- The unit circle in $\mathbb{R}^2$,
- the set of all integers,
- The set of non-negative functions on $[a, b]$.

Next, we would like to define a notion of distance between two elements of a vector space. The main tool for this is the concept of a metric.

Definition 2 (metric). *A **metric** on a set S is a function $\rho : S \times S \to \mathbb{R}$ such that for all elements $x, y, z \in S$,*

- $\rho(x, y) \geq 0$ *with equality if $x = y$.*
- *(symmetry)* $\rho(x, y) = \rho(y, x)$.
- *(triangular inequality)* $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

The interpretation of $\rho(x, y)$ is that it measures the 'distance' between the elements x and y in S. The first condition states that the distance between two elements of x and y is non-negative and zero if and only if x is equal to y. The second condition requires that the distance between x and y is the same as the distance between y and x. The final triangular condition encompasses the intuition that distance between x and y is no larger than the sum of the distance between x and a third element z plus the distance between z and y.[14] A set S together with a metric ρ on S is called a metric space, (S, ρ). Now,

[14] Going from x to y is always shorter than going from x to some third location z and then going from z to y.

The following are metric spaces:

- The set of integers with $\rho(x, y) = |x - y|$,
- The set of integers with $\rho(x, y) = 0$ if $x = y$ and 1 if $x \neq y$,
- The set of all continuous, strictly, increasing functions on $[a, b]$ with $\rho(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$.
- The set of all continuous, strictly increasing functions on $[a, b]$ with $\rho(x, y) = \int_a^b |x(t) - y(t)| dt$.

consider our previously defined vector space $C(X)$ of continuous real valued functions on a common domain X. Let us further restrict ourselves to the functions that are also bounded.[15] Let us call this the set $B(X)$. What would be a good metric on this set. In other words, if we take two bounded and continuous functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, how can we measure the 'distance' between these two functions?

A first idea would be to take one particular value $x_0 \in X$ and to define,

$$\rho(f, g) = |f(x_0) - g(x_0)|.$$

the problem with this 'metric', however, is that it does not satisfy the first condition: it is possible that $|f(x_0) - g(x_0)| = 0$, i.e. $f(x_0) = g(x_0)$, but f and g are distinct. This can be fixed by taking the maximal distance between f and g over the set X,

$$\rho(f, g) = \max_{x \in X} |f(x) - g(x)|.$$

The problem with this proposal is that the maximum may not exist (if for example X is not compact). We can solve this by taking the supremum instead of the maximum.[16]

$$\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

This metric is called the sup or infinity metric.[17] A closely related concept to a metric is a norm.

[15] A function $f : X \to \mathbb{R}$ is bounded if there exists a number $M > 0$ such that for all $x \in X$, $|f(x)| \leq M$. Observe that M is chosen independent of x.

[16] The sup exists because we assumed that both f and g are bounded.

[17] Show that it satisfies all three conditions to be a metric.

Definition 3 (normed vector space). *A **norm** of a vector space S is a function $\|.\| : S \to \mathbb{R}_+$ such that for all $x, y \in S$ and $\alpha \in \mathbb{R}$,*

- $\|x\| \geq 0$, *with equality if and only if* $x = \theta$,
- $\|\alpha x\| = |\alpha|\|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Let $\|.\|$ be a norm and define the metric $\rho(x, y) = \|x - y\|$. It is easy to verify that $\rho : S \times S \to \mathbb{R}$ defined in this way is a metric. As such, the number $\|x - y\|$ can be interpreted as measuring the distance between x and y. In this perspective $\|x\|$ measures the distance from x to the zero vector θ. A vector space S together with a norm $\|.\|$ is called a normed vector space. Looking back at our sup metric on the set $B(X)$ we see that we can define the following norm,

$$\|f\| = \rho(f, \theta) = \sup_{x \in X} |f(x) - 0| = \sup_{x \in X} |f(x)|.$$

This norm is called the sup or infinity norm.

The following are normed vector spaces:

- $\mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.
- $\mathbb{R}^n$ with $\|x\| = \max_i |x_i|$,
- $\mathbb{R}^n$ with $\|x\| = \sum_{i=1}^n |x_i|$.
- The set of all bonded infinite sequences $(x_1, \ldots)$ with $\|x\| = \sup_k |x_k|$ this space is called ℓ_∞.
- The set of continuous functions on $[a, b]$ with $\|x\| = \sup_{a \leq t \leq b} |x(t)|$ this space is called $C[a, b]$.
- The set of continuous functions on $[a, b]$ with $\|x\| = \int_a^b |x(t)| dt$.

THE MAIN REASON for introducing metrics or norms is to measure distance between different elements of a vector space. Once we can measure distances, we can also start talking about convergence.

Definition 4 (convergence). *Let $(S, \|.\|)$ be a normed vector space. A countable sequence $(x_n)_{n=0}^{\infty}$ of elements in S is said to **converge** to an element $x \in S$ if for all $\varepsilon > 0$, there exists a N_ε such that for all $n \geq N_\varepsilon$,*[18]

$$\|x_n - x\| < \varepsilon.$$

We also write this as $x_n \to x$.

[18] We write N_ε to make it clear that N_ε may be different for different values of ε.

In words, a sequence $\{x_n\}_{n=0}^{\infty}$ converges to an element x if for all strictly positive numbers ε, it is possible to go far enough in the sequence, say further than the N_ε'th element such that for all elements x_n beyond this element the distance between x_n and x is smaller than ε.[19]

[19] Alternatively, you could say that for any strictly positive number ε there are only a finite number of elements in the sequence $\{x_n\}_{n=0}^{\infty}$ that are at a distance greater than ε from x.

Next, we need the definition of a Cauchy sequence.

Definition 5 (Cauchy sequence). *Let $(S, \|.\|)$ be a normed vector space. A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a **Cauchy sequence** if for all $\varepsilon > 0$, there is a number N_ε such that for all $n, m \geq N_\varepsilon$,*

$$\|x_n - x_m\| < \varepsilon.$$

So a sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence if for any strictly positive number ε it is possible to go far enough in the sequence, further than N_ε such that the distance between any two elements beyond the N_ε'th position is less than ε.

Exercises:

- Show that if $x_n \to x$ and $x_n \to y$ then $x = y$.
- Show that if a sequence is convergent, then it satisfies the Cauchy criterion.
- Show that $x_n \to x$ if and only if every subsequence of $\{x_n\}$ converges to x.

Complete metric spaces

IT IS ALWAYS the case that a convergent sequence $x_n \to x$ in a normed vector space is also a Cauchy sequence. The reverse, however is not necessarily the case. In other words, it is possible that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, but it does not converge to an element in S. Vector spaces where this is true are called complete vector spaces.

Definition 6 (complete metric spaces). *A vector space $(S, \|.\|)$ is complete if every Cauchy sequence in S converges to an element in S.*

An example of a normed vector space that is not complete is the set of rational numbers $\mathbb{Q}$ with norm $|.|$. Consider for example the sequence,

Check that this is a normed vector space.

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

with starting value $x_0 = 1$. This gives the sequence,

$$\begin{aligned}&1\\&1.5\\&1.4166\ldots\\&1.4142156868\ldots\\&\ldots\end{aligned}$$

Each number in the sequence is rational. Additionally, this sequence has a as limit $x = \frac{x}{2} + \frac{1}{x}$ or $x = \sqrt{2}$. However, $\sqrt{2}$ is not a rational number. Another example of a normed vector space that is not complete is the open interval $(0, 1)$ with norm $|.|$. Here the sequence $\{1/n\}_{n=1}^{\infty}$ is Cauchy, but its limit 0 is not in the interval $(0, 1)$.

Intuitively, a complete vector space is a space without any 'points' missing, where the missing points could either ly inside or at its boundary. We take it as a fact that the set of real numbers $\mathbb{R}$ with the norm $|x - y|$ is a complete metric space.[20] A complete normed vector space is also called a **Banach space**. We will use the term Banach space from now on.

[20] This is a consequence of the Bolzano-Weierstrass theorem.

Let us return to our set of continuous bounded functions $B(X)$ on the common domain X. We know that this is a vector space and that $\|f\| = \sup_{x \in X} |f(x)|$ is a norm on this space. It is also possible to show that this is a Banach space.

For the analysis in the next chapters, it will however be useful to generalize the notion of the sup-norm. Let us go back to the set of continuous functions on the set X, which we denoted by $C(X)$. Let $\phi : X \to \mathbb{R}_{++}$ be a continuous function that takes only strictly positive values. For such given function ϕ, we consider the set of functions $B_\phi(X)$ for which the function $\frac{f(x)}{\phi(x)}$ is bounded. In other words, $f \in B_\phi(X)$ if there exists an M such that for all $x \in X$, $\frac{f(x)}{\phi(x)} \leq M$.

For these functions, we can consider the following norm,

$$\|f\|_\phi = \sup_{x \in X} \frac{|f(x)|}{\phi(x)}.$$

Let us first show that this is indeed a norm. First,

$$\|f\|_\phi \geq 0,$$

is easily established.[21] If $\|f\|_\phi = 0$. Then we have that for all $x \in X$,

$$0 = \frac{|f(x)|}{\phi(x)}.$$

[21] Indeed, both $|f(x)| \geq 0$ and $\phi(x) > 0$.

Given that $\phi(x) > 0$, we have indeed that $f(x) = 0$ for all $x \in X$, so f is the zero function. Next,

$$\|\alpha f\|_\phi = \sup_{x \in X} \frac{|\alpha f(x)|}{\phi(x)} = |\alpha| \sup_{x \in X} \frac{|f(x)|}{\phi(x)} = |\alpha| \|f\|_\phi.$$

finally,

$$\begin{aligned}\|f + g\|_\phi &= \sup_{x \in X} \frac{|f(x) + g(x)|}{\phi(x)},\\ &\leq \sup_{x \in X} \frac{|f(x)| + |g(x)|}{\phi(x)},\\ &\leq \sup_{x \in X} \frac{|f(x)|}{\phi(x)} + \sup_{x \in X} \frac{|g(x)|}{\phi(x)},\\ &= \|f\|_\phi + \|g\|_\phi.\end{aligned}$$

Observe that if we consider the constant function $\phi(x) = 1$ for all $x \in X$, then $\|f\|_\phi = \|f\|$. As such, the sup norm is a special case of the ϕ-norm with $\phi(x) = 1$ for all x. However, the ϕ-norm covers other cases to. Consider, for example $X = \mathbb{R}$ and $\phi(x) = |x| + 1$ then we see that f is bounded in the norm $\|.\|_\phi$, if f does not grow faster than $|x|$.[22] In other words, f can be unbounded but not 'more' unbounded than the function $\phi(x) = |x| + 1$. In general f will be bounded in the ϕ-norm if the value of $|f(x)|$ does not 'grow' faster than $\phi(x)$.

[22] For example, $f(x) = x^2$ is not bounded using this norm on $\mathbb{R}$. but $f(x) = ax + b$ is bounded although $f(x) = ax + b$ is not bounded in the sup-norm.

The following theorem shows that $B_\phi(X)$ is a Banach space.

Theorem 1. *Let $\phi : X \to R_{++}$ be a continuous function and let $B_\phi(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$ that are bounded in the norm $\|f\|_\phi = \sup_{x \in X} \frac{f(x)}{\phi(x)}$. Then $B_\phi(X)$ is a Banach space.*

Proof. That $B_\phi(X)$ is a normed vector space was shown above. Let $\{f_n\}$ be a Cauchy sequence in $B_\phi(X)$. We need to show that there exists an $f \in B_\phi(X)$ such that for all $\varepsilon > 0$ there is an N_ε such that for all $n > N_\varepsilon$,

$$\|f_n - f\|_\phi < \varepsilon.$$

There are three steps. First, we find a candidate function, second, we show that $\{f_n\}$ converges to this candidate function (in the $\|.\|_\phi$ norm). Third we show that the candidate function is in $B_\phi(X)$.

For step one, fix $x \in X$, then the sequence of real numbers $f_n(x)$ satisfies,

$$\begin{aligned}|f_n(x) - f_m(x)| &= \phi(x) \frac{|f_n(x) - f_m(x)|}{\phi(x)},\\ &\leq \phi(x) \sup_{y \in X} \frac{|f_n(y) - f_m(y)|}{\phi(y)} = \phi(x) \|f_n - f_m\|_\phi.\end{aligned}$$

As $\phi(x) > 0$, and $\{f_n\}$ is a Cauchy sequence, we see that $|f_n(x) - f_m(x)| \to 0$ as $n, m \to \infty$. As such, it satisfies the Cauchy criterion. As

$\mathbb{R}$ is complete, the sequence $\{f_n(x)\}$ has a limit, call it $f(x)$.[23] This defines a function $f : X \to \mathbb{R}$ that we take to be our candidate.

[23] This keeps x fixed and regards $\{f_n(x)\}$ as a sequence of numbers in $\mathbb{R}$.

For step 2, we need to show that $\|f_n - f\|_\phi \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Then there is a N_ε such that for all $n, m > N_\varepsilon$, $\|f_n - f_m\|_\phi < \varepsilon/3$. Take such $n > N_\varepsilon$. Now take any $x \in X$. Then there is an $N_{\varepsilon,x}$ such that for all $m \geq N_{\varepsilon,x}$, $|f_m(x) - f(x)| < (\varepsilon/3)\phi(x)$. Take any $m \geq \{N_\varepsilon, N_{\varepsilon,x}\}$. Then,

$$\begin{aligned}\frac{|f_n(x) - f(x)|}{\phi(x)} &\leq \frac{|f_n(x) - f_m(x)|}{\phi(x)} + \frac{|f_m(x) - f(x)|}{\phi(x)},\\ &\leq \|f_n - f_m\|_\phi + \frac{|f_m(x) - f(x)|}{\phi(x)},\\ &< \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3.\end{aligned}$$

This holds for all $x \in X$, so $\sup_{x\in X} \frac{|f_n(x)-f(x)|}{\phi(x)} = \|f_n - f\|_\phi \leq 2\varepsilon/3 < \varepsilon$.

Finally, we need to show that $f \in B_\phi(X)$. Boundedness of $\|f\|_\phi$ is obvious.[24] Let us first show that $\frac{f(x)}{\phi(x)}$ is continuous. As $\phi(x)$ is continuous, this also shows that $f(x)$ is continuous. So, let us show that if $x_t \to x$ then $f(x_t)/\phi(x_t) \to f(x)/\phi(x)$. Let $x_t \to x$. Fix a ε. Then there is a N_ε such that for all $n > N_\varepsilon$, $\|f_n - f\|_\phi < (\varepsilon/3)$.

[24] This follows from the fact that the sequence $\{f_n\}$ is a Cauchy sequence.

Take any such f_n. Then, as f_n is continuous, there is a N_2 such that for all $t > N_2$, $\left|\frac{f_n(x_t)}{\phi(x_t)} - \frac{f_n(x)}{\phi(x)}\right| < \varepsilon/3$. Then,

$$\begin{aligned}\left|\frac{f(x_t)}{\phi(x_t)} - \frac{f(x)}{\phi(x)}\right| &= \left|\frac{f(x_t)}{\phi(x_t)} - \frac{f_n(x_t)}{\phi(x_t)} + \frac{f_n(x_t)}{\phi(x_t)} - \frac{f_n(x)}{\phi(x)} + \frac{f_n(x)}{\phi(x)} - \frac{f(x)}{\phi(x)}\right|,\\ &\leq \frac{|f(x_t) - f_n(x_t)|}{\phi(x_t)} + \left|\frac{f_n(x_t)}{\phi(x_t)} - \frac{f_n(x)}{\phi(x)}\right| + \frac{|f_n(x) - f(x)|}{\phi(x)},\\ &< \|f_n - f\|_\phi + \varepsilon/3 + \|f_n - f\|_\phi < \varepsilon.\end{aligned}$$

□

When we take $\phi(x) = 1$ for all x, this theorem shows that $B(X)$ being the set of all continuous functions that are bounded in the $\|.\|$ norm.[25] is also a Banach space.

[25] These are the continuous functions that are simply bounded.

Corollary 1. *Let $X \subseteq \mathbb{R}^n$ and let $B(X)$ be the set of all bounded continuous functions $f : X \to \mathbb{R}$ with the sup norm $\|f\| = \sup_{x\in X} |f(x)|$. Then $B(X)$ is a Banach space.*

Contraction mappings

Now that we are equipped with the notion of a Banach space, we can have a look at contraction mappings.

Definition 7 (contraction mapping)**.** *Let $(S, \|.\|)$ be a normed vector space and let $T : S \to S$ be a function mapping S into itself. The operator T is a **contraction mapping** with modulus $\beta \in (0,1)$ if*

$$\|Tx - Ty\| \leq \beta \|x - y\|.$$

A function is a contraction mapping if the distance between the two images of points x and y are closer together than the original points x and y. Intuitively, when we iterate such mapping, the points will at each step come closer and closer together. Eventually, we expect these iterations to converge to what we call a **fixed point**.

Definition 8. *Let $(S, \|.\|)$ be a metric space and let T be a function from S to S. Then $x \in S$ is called a fixed point of T if*

$$Tx = x.$$

Let $B_\phi(X)$ be our set of continuous functions on X that are bounded in the ϕ-norm. A mapping T from $B_\phi(X)$ to $B_\phi(X)$ takes a function $f \in B_\phi(X)$ as inputs and produces another function $g = (Tf) \in B_\phi(X)$. A fixed point of T is a function $f \in B_\phi(X)$ such that T maps f to itself $f = (Tf)$. A function T that takes functions to functions is called, for clarity, an operator. Every contraction mapping (operator) on a Banach space has a unique fixed point.

Theorem 2 (Banach's contraction mapping theorem)**.** *Let $(S, \|.\|)$ is a Banach space and let $T : S \to S$ be a contraction mapping on S with modulus β, then*

- *T has exactly one fixed point $v \in S$,*
- *For any $v_0 \in S$, $\|(T^n v_0) - v\| \leq \beta^n \|v_0 - v\|$.*

Proof. Define the iterates of T, the mappings $\{T^n\}$ by $T^0 x = x$, $T^n x = T(T^{n-1}x)$. Choose $v_0 \in S$ and let $\{v_n\}_{n=1}^\infty$ be defined as $v_n = T^n v_0$. By the contraction mapping property,

$$\|v_2 - v_1\| = \|Tv_1 - Tv_0\| \leq \beta \|v_1 - v_0\|.$$

By induction, we can show that,

$$\|v_{n+1} - v_n\| \leq \beta^n \|v_1 - v_0\|.$$

As such, for any $m \geq n$,

$$\begin{aligned}
\|v_m - v_n\| &\leq \|v_m - v_{m-1}\| + \|v_{m-1} - v_{m-2}\| + \ldots + \|v_{n+1} - v_n\|, \\
&\leq \left[\beta^{m-1} + \ldots, + \beta^{n+1} + \beta^n\right] \|v_1 - v_0\|, \\
&= \beta^n \left[\beta^{m-n-1} + \ldots + \beta + 1\right] \|v_1 - v_0\|, \\
&\leq \frac{\beta^n}{1-\beta} \|v_1 - v_0\|.
\end{aligned}$$

This shows that $\{v^n\}$ is a Cauchy sequence, so it has a limit $v_n \to v \in S$. To show that $Tv = v$ note that

$$\begin{aligned}\|Tv - v\| &\leq \|Tv - T^n v_0\| + \|T^n v_0 - v\|,\\ &\leq \beta\|v - T^{n-1}v_0\| + \|T^n v_0 - v\|,\end{aligned}$$

Both terms on the right hand side converge to zero. Hence $\|Tv - v\| = 0$, or $Tv = v$. For uniqueness, assume that $v, \hat{v}$ are both fixed points of T, then

$$\|\hat{v} - v\| = \|T\hat{v} - Tv\| \leq \beta\|\hat{v} - v\|.$$

This can only be true if $\|\hat{v} - v\| = 0$ or $\hat{v} = v$. □

It will often be convenient to restrict the region in the set S where the fixed point is situated. Let $(S, \|.\|)$ be a Banach space and let S' be a closed subset of S. It can be shown that the smaller set $(S', \|.\|)$ is also a Banach space.[26] If $T : S \to S$ is a contraction mapping and if T maps S' to S' then T is also contraction mapping on the smaller set $(S', \|.\|)$.[27] As S' is closed, the unique fixed point of T will lie in S'. This is the gist of the following lemma.

[26] A set S' is closed if for all sequences $\{x_t\}$ in S', $x_t \to x$ (according to the norm $\|.\|$) implies that $x \in S'$.

[27] This requires that $T(S') \subseteq S'$.

Lemma 1. *Let $(S, \|.\|)$ be a Banach space and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $v \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$ then $v \in S''$.*

Proof. Choose $v_0 \in S'$ and note that $\{T^n v_0\}$ is a sequence in S' converging to the fixed point v of T. Since S' is closed, it follows that $v \in S'$, so the unique fixed point is also in S'. If in addition $T(S') \subseteq S''$ then it follows that $v = Tv \in S''$ so v is also in S''. □

The second part of the contraction mapping theorem provides a bound on the distance from the n-th iterate $T^n v_0$ to the fixed point v,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|.$$

This bound, however, is not very useful as it involves the 'unknown' function v. The following gives a computationally more relevant bound.

Lemma 2. *Let $(S, \|.\|)$ be a Banach space, T a contraction mapping and v the fixed point of T. Then*

$$\|T^n v_0 - v\| \leq \frac{1}{1-\beta}\|T^n v_0 - T^{n+1}v_0\|.$$

Proof. Notice that,

$$\begin{aligned}\|T^n v_0 - v\| &\leq \|T^n v_0 - T^{n+1}v_0\| + \|T^{n+1}v_0 - v\|,\\ &\leq \|T^n v_0 - T^{n+1}v_0\| + \beta\|T^n v_0 - v\|.\end{aligned}$$

Rearranging this inequality gives the desired result. □

PREVIOUSLY, WE SAW that the set of continuous functions $f : X \to \mathbb{R}$ that are bounded in the $\|.\|_\phi$ norm, i.e. $B_\phi(X)$ was a Banach space. We will mainly be interested in contraction mappings from $B_\phi(X) \to B_\phi(X)$. These contraction mappings take functions in $B_\phi(X)$ to other functions in $B_\phi(X)$. The following theorem, known as Blackwell's theorem provides sufficient, easy to verify, conditions for an operator T to be a contraction mapping on $B_\phi(X)$.

Theorem 3 (Blackwell's sufficient conditions). *Let $\phi : X \to R_{++}$ be a continuous function and let let $B_\phi(X)$ be the space of continuous functions $f : X \to \mathbb{R}$, that are bounded in the norm $\|.\|_\phi$. Let $T : B_\phi(X) \to B_\phi(X)$ be an operator satisfying,*

- *(monotonicity) If $f, g \in B_\phi(X)$ and $f(x) \le g(x)$ for all $x \in X$, then $(Tf)(x) \le (Tg)(x)$ for all $x \in X$.*
- *(discounting) for all $x \in X$, $f \in B_\phi(X)$ and $a \ge 0$, there is some $\beta \in (0, 1)$ such that,*

$$T(f + a\phi)(x) \le (Tf)(x) + (\beta a)\phi(x),$$

Then T is a contraction with modulus β.

Proof. Observe that

$$\begin{aligned}\frac{f(x)}{\phi(x)} &= \frac{g(x)}{\phi(x)} + \frac{f(x) - g(x)}{\phi(x)},\\ &\le \frac{g(x)}{\phi(x)} + \frac{|f(x) - g(x)|}{\phi(x)},\\ &\le \frac{g(x)}{\phi(x)} + \|f - g\|_\phi.\end{aligned}$$

as such, multiplying both sides by $\phi(x) > 0$ gives,

$$f(x) \le g(x) + \phi(x)\|f - g\|_\phi.$$

So, by monotonicity

$$(Tf)(x) \le T(g + \|f - g\|_\phi \phi)(x) \le (Tg)(x) + \beta\|f - g\|_\phi \phi(x).$$

Equivalently,

$$\frac{(Tf)(x) - (Tg)(x)}{\phi(x)} \le \beta\|f - g\|_\phi.$$

Reversing the roles of f and g gives

$$\frac{(Tg)(x) - (Tf)(x)}{\phi(x)} \le \beta\|f - g\|_\phi.$$

This holds for all x and the right hand side does not depend on x, so

$$\|Tf - Tg\|_\phi \le \beta\|f - g\|,$$

so T is a contraction mapping with modulus β. □

Applying above theorem to the case $\phi(x) = 1$,[28] we obtain the following, better known, version of Blackwell's theorem.

[28] This is the case where $\|.\|_\phi$ is the sup norm and $B(X)$ is the set of bounded continuous functions on X.

Corollary 2 (Blackwell's sufficient conditions)**.** *Let $X \subseteq \mathbb{R}^l$ and let $B(X)$ be the space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying,*

- *(monotonicity) If $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*
- *(discounting) for all $x \in X$, there is some $\beta \in (0,1)$ such that,*

$$T(f + a)(x) \leq (Tf)(x) + \beta a,$$

for all $f \in B(X)$ and $a \geq 0$.

Then T is a contraction with modulus β.

Theorem of the maximum

In the second part of this chapter we'll have a look at a seminal result in economics, the theorem of the maximum. Consider tow sets $X \subseteq \mathbb{R}^l$, and $A \subseteq \mathbb{R}^m$, and let $f : X \times A \to \mathbb{R}$ be a real valued function that takes a vector $x \in X$, a vector in $y \in A$ and produces a real number $f(x, y)$. Let $\Gamma : X \to A$ be a correspondence.[29] The theorem of the maximum deals with optimization problems of the following form,

[29] A correspondence $\Gamma : X \to A$ is a mathematical object that takes a vector $x \in X$ and delivers a subset $\Gamma(x) \subseteq A$.

$$\max_{y \in \Gamma(x)} f(x, y).$$

This problem optimizes a function $f(x, y)$ with respect to y, when y is restricted to lie in the set $\Gamma(x)$. Here x is kept fixed, so it is a parameter of the optimization problem. If for each x, $f(x, .)$ is continuous in y and the set $\Gamma(x)$ is nonempty and compact,[30] then for all x the maximum is attained.[31] In this case, we can define the function

[30] Here, compactness means that for each $x \in X$, $\Gamma(x)$ is closed and bounded.

[31] This follows from the extreme value theorem.

$$v(x) = \max_{y \in \Gamma(x)} f(x, y),$$

and the set,

$$G(x) = \{y \in \Gamma(x) : f(x, y) = v(x)\},$$

of values in $\Gamma(x)$ that attain this maximum. We would like to place additional restrictions such that the function v and the set G vary continuously with the 'parameter' x.

Towards this end, we need to define the concepts of lower and upper hemi-continuity.

Definition 9 (Lower hemi-continuity)**.** *The correspondence $\Gamma : X \to A$ is **lower hemi-continuous** (l.h.c.) at $x \in X$ if*

1. *$\Gamma(x)$ is non-empty*

2. *for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exists an $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(y_n)$ for all $n \geq N$.*[32]

[32] If $\Gamma(y_n)$ is nonempty for all n, we can always take $N = 1$.

Definition 10 (Upper hemi-continuity). *A compact-valued correspondence $\Gamma : X \to A$ is **upper hemi-continuous** (u.h.c.) at $x \in X$ if*

1. *$\Gamma(x)$ is non-empty, closed and bounded.*

2. *for every sequence $x_n \to x$ and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$.*

Definition 11 (continuity). *A correspondence $\Gamma : X \to A$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c. at x. A correspondence is continuous if it is continuous at each point in its domain.*

The following lemma is a well known result concerning convergence of sequences and will be useful in the proof of the following theorem.

Lemma 3. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. The function v is continuous at x if and only if for all sequences $\{x_n\}_{n=0}^{\infty}$ with $x_n \to x$ there is a subsequence $x_{n_t} \to x$ such that $v(x_{n_t}) \to v(x)$.*

Proof. ($\to$) Let v be continuous and $x_n \to x$. Then evidently $v(x_n) \to v(x)$ so for all subsequences $x_{n_t} \to x$ of $\{x_n\}$ we should have that $v(x_{n_t}) \to v(x)$.

($\leftarrow$) For the reverse, assume that v is not continuous at x. Then there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $v(x_n) \not\to v(x)$. From this, we will construct a sequence $z_t \to x$ that has no subsequence z_{t_n} such that $v(z_{t_n})$ converges to $v(x)$.

As $v(x_n) \not\to v(x)$, there exists a $\varepsilon > 0$ such that for all T, there is an $n_T \geq T$ such that $|v(x) - (v(x_{n_T})| > \varepsilon$. Let $n_1, n_2, n_3, \ldots$ be this sequence of numbers. Consider the following subsequence of $\{z_n\}_{n=0}^{\infty}$.

- set $z_1 = x_0$.
- let $z_2 = x_{n_1}$.
- pick $T_1 \geq n_1$ and set $z_3 = x_{n_{T_1}}$,
- pick $T_2 \geq n_{T_1}$ and set $z_4 = x_{n_{T_2}}$,
- pick $T_3 \geq n_{T_2}$ and set $z_5 = x_{n_{T_3}}$,
- ...

We see that $\{z_t\}_{t=0}^{\infty}$ is a sequence such that $z_t \to x$. Additionally, $\{v(z_t)\}_{t=0}^{\infty}$ has no subsequence $\{v(z_{t_n})\}$ that converges to $v(x)$, as was to be shown. □

The graph of a correspondence $\Gamma : X \to X$ is given by $A = \{(x,y) : y \in \Gamma(x)\}$. It is easy to see that Γ is u.h.c. if the graph A is compact.

Theorem 4 (Theorem of the maximum)**.** *Let $X \subseteq \mathbb{R}^l, A \subseteq \mathbb{R}^m$. Let $f : X \times A \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to X$ be a compact-valued and continuous correspondence. Then the function $v : X \to \mathbb{R}$,*

$$v(x) = \max_{y \in \Gamma(x)} f(x,y).$$

is continuous, and the correspondence $G : X \to X$

$$G(x) = \{y \in \Gamma(x) : f(x,y) = v(x)\},$$

is non-empty, compact valued and u.h.c.

Proof. Let us first show that for all $x \in X$, $G(x)$ is non-empty and compact valued. Fix an element $x \in X$. The set $\Gamma(x)$ is nonempty and compact and $f(x,.)$ is continuous. From the extreme value theorem, we know that the maximum is attained and the set $G(x)$ is non-empty. Since $G(x) \subseteq \Gamma(x)$ and $\Gamma(x)$ is bounded, we have that $G(x)$ is also bounded. We still need to show that $G(x)$ is closed. Let $y_n \to y$ and $y_n \in G(x)$ for all n. Since $\Gamma(x)$ is u.h.c., we know that $y \in \Gamma(x)$. Also, since $v(x) = f(x, y_n)$ for all n and f is continuous, $\lim_n f(x,y_n) = f(x,y) = v(x)$. As such, $y \in G(x)$ which shows that $G(x)$ is closed. Conclude that $G(x)$ is compact valued.

Next, we show that v is continuous. By the above lemma, it suffices to show that any sequence $x_n \to x$ has a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ such that $v(x_{n_k}) \to v(x)$. Take any $x \in X$ and any sequence $\{x_n\}$ be a sequence in X converging to x. We need to construct a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ such that $v(x_{n_k}) \to v(x)$.

As $x_n \to x$ we have for all n and element $y_n \in G(x_n)$ and $v(x_n) = f(x_n, y_n)$. As Γ is u.h.c. we have that there exists a subsequence $\{y_{n_k}\}$ converging to $y \in \Gamma(x)$. Also, as f is continuous, $\lim_k f(x_{n_k}, y_{n_k}) = \lim_k v(x_{n_k}) = f(x,y)$. Let us finish the proof by showing that $f(x,y) = v(x)$. If not, then there is an element $y' \in G(x)$ such that $v(x) = f(x,y') > f(x,y)$. Consider a number $\varepsilon > 0$ such that $f(x,y') > f(x,y) + \varepsilon$.

Also $y' \in \Gamma(x)$, so by l.h.c. we have that there is a sequence $y'_{n_k} \to y'$ such that $y'_{n_k} \in \Gamma(x_{n_k})$. By continuity of f, we can take n_k large enough such that $|f(x,y') - f(x_{n_k}, y'_{n_k})| < \varepsilon/2$. Additionally, we can take n_k large enough such that additionally, $|f(x,y) - f(x_{n_k}, y_{n_k})| < \varepsilon/2$. But then,

$$f(x_{n_k}, y'_{n_k}) > f(x,y') - \varepsilon/2 > f(x,y) + \varepsilon/2 > f(x_{n_k}, y_{n_k}).$$

However, this contradicts the assumption that y_{n_k} was optimal for x_{n_k}, i.e., $f(x_{n_k}, y_{n,k}) = v(x_{n_k}) \geq f(x_{n_k}, y'_{n_k})$.

Next, let us show that $G(x)$ is u.h.c. Fix x and let $\{x_n\}$ be a sequence converging to x. Let $y_n \in G(x_n)$. Since Γ is u.h.c., there exists a subsequence $\{y_{n_k}\}$ converging to some $y \in \Gamma(x)$. We need to show that $y \in G(x)$. By continuity of v and f.

$$v(x) = \lim_n v(x_{n_k}) = \lim_n f(x_{n_k}, y_{n_k}) = f(x, y).$$

As such, $y \in G(x)$ which shows that G is u.h.c. □

In general the optimal value correspondence is not l.h.c. Consider the example where $X = \mathbb{R}$, $f(x, y) = xy^2$ and $\gamma(x) = [-1, 1]$ for all x. Then

$$G(x) = \begin{cases} \{-1, 1\} \text{ if } x > 0 \\ [-1, 1] \text{ if } x = 0, \\ 0 \text{ if } x < 0. \end{cases}$$

Take a sequence $x_t \rightarrow 0$ where $x_t < 0$ for all t. Then $0.5 \in G(0)$ but $G(x_t) = \{-1, 1\}$ for all x_t so there is not sequence in $G(x_t)$ that converges to 0.5 which means that the correspondence G is not l.h.c. at $x = 0$.

Theorem 5. *Let $X \subseteq \mathbb{R}^l$. Let $\Gamma : X \rightarrow X$ be convex valued, compact valued and continuous. Assume that $f : A \rightarrow \mathbb{R}$ is continuous and that $f(x, .)$ is strictly concave in its second argument then if*

$$v(x) = \max_{y \in \Gamma(x)} f(x, y),$$

we have that

$$g(x) = \{y \in \Gamma(x) : f(x, y) = v(x).\}$$

is single valued, and the function $g(x)$ is continuous.

Proof. From the theorem of the maximum, we know that G is bounded, compact valued and u.h.c. Let us first show that G is single valued. Assume, towards a contradiction, that $y_1, y_2 \in G(x)$. Then $\theta y_1 + (1 - \theta) y_2 \in \Gamma(x)$ for $\theta \in (0, 1)$ so,

$$f(x, \theta y_1 + (1 - \theta) y_2) > \theta f(x, y_1) + (1 - \theta) f(x, y_2) = v(x).$$

but this contradicts the optimality of y_1 and y_2. This shows that g is a single valued function. For continuity, let $x_n \rightarrow x$. It suffices to show that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $g(x_{n_k}) \rightarrow g(x)$. Let $y_n \in g(x_n)$. Then by u.h.c. of g there is a subsequence y_{n_k} such that $g(x_{n_k}) = y_{n_k} \rightarrow y = g(x)$ as was to be shown. □

The second part of the theorem is actually true in general. If g is a function and u.h.c. then it is continuous. Also if g is a function and l.h.c. then it is also continuous.

Theorem 6. *Let $X \subseteq \mathbb{R}^l$. Let $\Gamma : X \rightarrow X$ be convex valued, compact valued and continuous. Let $\{f_n\}$ be a sequence of continuous functions on A and assume that for all n, $f_n(x, .)$ is strictly concave. Assume that f has the same properties and that $\|f_n - f\|_\phi \rightarrow 0$. Let,*

$$g_n = \arg\max_{a \in \Gamma(x)} f_n(x, a),$$

and

$$g = \arg\max_{a \in \Gamma(x)} f(x, a).$$

then for all x, $g_n(x) \rightarrow g(x)$. If X is compact then $\|g_n(x) - g(x)\|_\phi \rightarrow 0$.

Proof. We have that,

$$
\begin{aligned}
0 &\leq f(x, g(x)) - f(x, g_n(x)), \\
&\leq (f(x, g(x)) - f(x, g_n(x))) + \underbrace{(f_n(x, g_n(x)) - f_n(x, g(x)))}_{\geq 0}, \\
&= (f(x, g(x)) - f_n(x, g(x)) + (f_n(x, g_n(x)) - f(x, g_n(x))), \\
&\leq \phi(x)\|f - f_n\|_\phi + \phi(x)\|f_n - f\|_\phi \to 0.
\end{aligned}
$$

As such, we have that for all $x \in X$,

$$\left| \frac{f(x, g(x)) - f(x, g_n(x))}{\phi(x)} \right| \leq 2\|f - f_n\|_\phi.$$

This shows that,

$$\sup_{x \in X} \left| \frac{f(x, g(x)) - f(x, g_n(x))}{\phi(x)} \right| \to 0.$$

First for pointwise convergence, assume, towards a contradiction that for some x, $g_n(x) \not\to g(x)$. Then there is a subsequence $g_{n_k}(x)$ and a $\varepsilon > 0$ such that for all n_k,

$$|g_{n_k}(x) - g(x)| \geq \varepsilon.$$

Let,

$$A_\varepsilon = \{y \in \Gamma(x) : |y - g(x)| \geq \varepsilon\}.$$

We know that for all n_k, $g_{n_k}(x) \in A_\varepsilon$, so it is non-empty. Also the element $y = g(x)$ is not in A_ε. The set A_ε is also compact.

Let

$$\delta = \min_{y \in A_\varepsilon} |f(x, g(x)) - f(x, y)|.$$

This problem is well defined as A_ε is a compact set. Also $f(x, g(x)) \geq f(x, y)$ for all $y \in A_\varepsilon \subseteq \Gamma(x)$. In fact, the absolute value $|f(x, g(x)) - f(x, y)|$ is equal to 0 only if y solves the maximization problem which means that in this case $g(x) = y$. However $y = g(x)$ is not in A_ε which means that for minimization problem $\delta > 0$. As $g_n(x) \in A_\varepsilon$ it follows that for all n_k,

$$|f(x, g(x)) - f(x, g_{n_k}(x))| > \delta.$$

However $|f(x, g(x)) - f(x, g_{n_k}(x))| \to 0$ as $\sup_{x \in X} \left| \frac{f(x, g(x)) - f(x, g_n(x))}{\phi(x)} \right| \to 0$, a contradiction.

Now, let X be compact. Let us show that $\|g_n - g\|_\phi \to 0$. If not then there exists a subsequence g_{n_k} such that for all n_k,

$$\|g_{n_k} - g\|_\phi \geq \varepsilon.$$

In particular, for all n_k there exists an $x \in X$ such that

$$\frac{\|g_{n_k}(x) - g(x)\|}{\phi(x)} \geq \varepsilon.$$

Let,

$$A_\varepsilon = \left\{ (x,y) \in X \times X : y \in \Gamma(x), \frac{\|y - g(x)\|}{\phi(x)} \geq \varepsilon \right\}.$$

Again, we see that A_ε is compact. For all n_k, there is an x such that $(x, g_{n_k}(x)) \in A_\varepsilon$, so it is non-empty and for all x, $(x, g(x)) \notin A_\varepsilon$.

Let,

$$\delta = \min_{(x,y) \in A_\varepsilon} \frac{|f(x,g(x)) - f(x,y)|}{\phi(x)}.$$

Observe that the objective function is zero only if $f(x, g(x)) = f(x, y)$ which only happens if $g(x) = y$. As such, $\delta > 0$. This means that for all n_k there is an x such that,

$$\frac{|f(x,g(x)) - f(x,g_n(x)|}{\phi(x)} \geq \delta > 0.$$

This contradicts the fact that,

$$\sup_{x \in X} \left| \frac{f(x,g(x)) - f(x,g_n(x))}{\phi(x)} \right| \to 0.$$

□

Dynamic programming under certainty

In this chapter we will investigate the infinite horizon optimization problem that we presented in chapter 1.

We denote by $X \subseteq \mathbb{R}^l$ the state space. An element $x \in X$ captures the state of the world at a particular point in time. We denote by $A \subseteq \mathbb{R}^m$ the set of controls variables and we denote by $\Gamma : X \to A$ the correspondence that determines for all states x the possible values of the control variable. The next period feasible states are determined by a function $r : X \times A \to X$ such that $y = r(x, a) \in X$ is the next period's state if the current state is x and the chosen control variable has the value a. We also denote by $F : X \times A \to \mathbb{R}$ the instantaneous payoff function that depends on the values of the current state and control. Finally, let $\beta \in (0, 1)$ be a discount factor. In this section, we will be interested in finding solutions to the following infinite horizon optimization problem,

$$\begin{aligned} \max_{\{a_0, a_1, a_2, \ldots\}} & \sum_{t=0}^{\infty} \beta^t F(x_t, a_t), \\ \text{s.t. } & x_{t+1} = r(x_t, a_t), \\ & a_t \in \Gamma(x_t), \\ & x_0 \text{ given.} \end{aligned}$$

We will do this by relating it to the fixed point of the so called Bellman operator, $T : B_\phi(X) \to B_\phi(X)$ defined by,

$$(Tv)(x) = \max_{a \in \Gamma(x)} \{F(x, a) + \beta v(r(x, a))\}.$$

In particular, we will show that under certain conditions the first problem has a solution and its solution is equivalent to the fixed point of the Bellman operator Tv.

Definition 12 (Regularity condition)**.** *The problem (X, A, Γ, F, β) is **regular** if the one period return function $F : X \times A \to \mathbb{R}$ and the transition function $r : X \times A \to \mathbb{R}$ are continuous, if the transition correspondence $\Gamma : X \to A$ is non-empty, continuous and compact valued and, additionally, there is a continuous function $\phi : X \to \mathbb{R}_{++}$ such that,*

1. *There exists an $M \geq 0$ such that for all $x \in X$,*

$$\max_{a\in\Gamma(x)} |F(x,a)| \leq M\phi(x).$$

2. *There exists a $\theta \in (0,1)$ such that for all $x \in X$,*

$$\beta \max_{a\in\Gamma(x)} \phi(r(x,a)) \leq \theta\phi(x).$$

Observe that if the function F is bounded, then the regularity conditions are satisfied by choosing $\phi(x) = 1$ for all x.

Theorem 7. *If the problem (X, A, Γ, F, β) is regular then the Bellman operator is a contraction mapping from $B_\phi(X)$ to $B_\phi(X)$.*

Proof. If $v \in B_\phi(X)$, then v is continuous. Also F and r are continuous and Γ is continuous and compact valued. As such, the optimization problem,

$$(Tv)(x) = \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\},$$

is well defined for all $x \in X$. By the theorem of the maximum, the maximum value function is continuous in x. As such, $(Tv)(x)$ is a continuous function of x. in order to show that $T : B_\phi(X) \to B_\phi(X)$ it suffices to show that $\|Tv\|_\phi$ is bounded whenever $\|v\|_\phi$ is bounded. Now, assume that $\|v\|_\phi < N$ then,

$$\begin{aligned}
(Tv)(x) &= \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\},\\
&\leq \max_{a\in\Gamma(x)} F(x,a) + \beta \max_{a\in\Gamma(x)} v(r(x,a)),\\
&\leq M\phi(x) + \beta N \max_{a\in\Gamma} \phi(r(x,a)),\\
&\leq M\phi(x) + N\theta\phi(x) = (M + N\theta)\phi(x).
\end{aligned}$$

This shows that $\|Tv\|_\phi \leq M + N\theta$, so $\|Tv\|_\phi$ is bounded. Conclude that $Tv \in B_\phi(X)$.

In order to show that T is a contraction mapping, we use Blackwell's theorem. For monotonicity, assume that $v \geq w$. Then,

$$\begin{aligned}
(Tv)(x) &= \max_{a\in\Gamma(x)} F(x,a) + \beta v(r(x,a)),\\
&\geq \max_{a\in\Gamma(x)} F(x,a) + \beta w(r(x,a)) = (Tw)(x).
\end{aligned}$$

For additivity,

$$\begin{aligned}
(T(v + a\phi))(x) &= \max_{a\in\Gamma(x)} \{F(x,a) + \beta\,(v + a\phi)\,(r(x,a))\},\\
&\leq \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\} + a \max_{a\in\Gamma(x)} \beta\phi(r(x,a)),\\
&\leq (Tv)(x) + a\theta\phi(x).
\end{aligned}$$

□

Above theorem shows that the Bellman operator has a fixed point, say v, which is continuous and bounded in the $\|.\|_\phi$ norm. Associated with the Bellman operator, we can find a policy correspondence, G where

$$G(x) = \{a \in \Gamma(x) : F(x,a) + \beta v(r(x,a)) = v(x)\}.$$

Consider our infinite horizon optimization problem.

$$\begin{aligned} \sup_{(a_t)_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(x_t, a_t), \\ \text{s.t. } & a_t \in \Gamma(x_t), \\ & x_{t+1} = r(x_t, a_t), \\ & x_0 \in X \text{ given.} \end{aligned}$$

When is this problem well defined? When can we replace the sup with a max operator? When does the solution coincide with the fixed point of the Bellman operator? These are three questions that we are going to answer now.

Definition 13 (feasible path)**.** *A* ***feasible path*** *is a sequence* $\{(x_0, a_0), (x_1, a_1), \ldots\}$ *such that for all* $t \in \mathbb{N}$*,*

1. $a_t \in \Gamma(x_t)$,
2. $x_{t+1} = r(x_t, a_t)$.

Let $\Pi(x_0)$ *be the set of all feasible paths that start at the state* $x_0 \in X$*. Then for such paths* $p = \{(x_0, a_0), (x_1, a_1), \ldots, \} \in \Pi(x_0)$ *we define,*

$$w_p = \sum_{t=0}^{\infty} \beta^t F(x_t, a_t) \equiv \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, a_t),$$

whenever this limit is well defined.

Lemma 4. *Assume that* (X, Γ, F, β) *is regular and let* $x_0 \in X$*. Then for all paths* $p \in \Pi(x_0)$*,* w_p *is well defined and the set* $\{w_p : p \in \Pi(x_0)\}$ *is bounded from above.*

Proof. Fix a feasible path $p = \{(x_0, a_0), (x_1, a_1), \ldots, \} \in \Pi(x_0)$ and define $u_n = \sum_{t=0}^{n} \beta^t F(x_t, a_t)$. In order to show that π_p is well defined, we need to show that the sequence $\{u_1, u_2, \ldots\}$ converges. We do this by showing that $\{u_n\}$ is a Cauchy sequence.[33] Fix $\varepsilon > 0$. We have to find an $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$.

$$|u_n - u_m| < \varepsilon.$$

[33] Remember that $\mathbb{R}$ is a Banach space, so every Cauchy sequence of real numbers converges.

Assume w.l.o.g. that $n > m$ then,

$$\begin{aligned} |u_n - u_m| &= \left| \sum_{t=m}^{n} \beta^t F(x_t, a_t) \right|, \\ &\le \sum_{t=m}^{n} \beta^t |F(x_t, a_t)|, \\ &\le \sum_{t=m}^{n} M\beta^t \phi(x_t). \end{aligned}$$

Now, $\phi(x_t) \le \max_{a\in\Gamma(x_{t-1})} \phi(r(x_{t-1}, a)) \le \frac{\theta}{\beta}\phi(x_{t-1})$. Iterating this further, we see that, $\phi(x_t) \le \left(\frac{\theta}{\beta}\right)^t \phi(x_0)$. As such,

$$\begin{aligned} |u_n - u_m| &\le \sum_{t=m}^{n} M\theta^t \phi(x_0), \\ &\le M\theta^m \phi(x_0) \sum_{t=0}^{n-m} \theta^t, \\ &\le M\theta^m \frac{\phi(x_0)}{1-\theta}. \end{aligned}$$

The right hand side goes to zero for $m \to \infty$. As such, $\{u_n\}$ is a Cauchy sequence, which means that it has a limit.

Now let us show that there is an A (which may depend on x_0) such that for all paths $p = \{(x_0, a_0), (x_1, a_1), \ldots\} \in \Pi(x_0)$, $w_p \le A$.[34] As above, we have that,

[34] Remember: every set of real numbers which is bounded from above has a supremum, so this shows that $\sup\{w_p : p \in \Pi(x_0)\}$ is well defined.

$$\beta^t |F(x_t, a_t)| \le M\beta^t \phi(x_0).$$

This gives,

$$\begin{aligned} u_n &= \sum_{t=1}^{T} \beta^t F(x_t, a_t), \\ &\le \sum_{t=1}^{T} \beta^t |F(x_t, a_t)|, \\ &\le \sum_{t=1}^{T} \beta^t \theta M \phi(x_0), \\ &\le \theta M \phi(x_0) \frac{1}{1-\beta}. \end{aligned}$$

Setting $A = \theta M\phi(x_0)/(1-\beta)$ demonstrates the proof. □

This lemma shows that $V(x_0) = \sup\{w_p : p \in \Pi(x_0)\}$ is well defined.

The next step is to show that the fixed point of the Bellman operator v satisfies for all $x_0 \in X$, (i) $v(x_0) \ge V(x_0)$ and (ii) for all $x_0 \in X$, there is a path $p \in \Pi(x_0)$ such that $v(x_0) = w_p$.

Lemma 5. *Let* (X, A, Γ, F, β) *be a regular problem and let* v *be the fixed point of the Bellman operator. Then for all paths* $p \in \Pi(x_0)$ $v(x_0) \geq w_p$.[35]

[35] In other words, $v(x_0)$ is an upper bound for $\{w_p : p \in \Pi(x_0)\}$.

Proof. Let v be the fixed point of the Bellman operator and let $p = \{(x_0, a_0), (x_1, a_1), \ldots\} \in \Pi(x_0)$ be a path. We will show that $v(x_0) \geq w_p$.

Now, by definition $v(x) = \max_{a \in \Gamma(x)} F(x, a) + \beta v(r(x, a))$, so,

$$\begin{aligned} v(x_0) &\geq F(x_0, a_0) + \beta v(x_1), \\ &\geq F(x_0, a_0) + \beta F(x_1, a_1) + \beta^2 v(x_2), \\ &\ldots \\ &\geq \sum_{t=0}^{T} \beta^t F(x_t, a_t) + \beta^T v(x_T). \end{aligned}$$

Taking limits, $T \to \infty$, gives[36],

$$v(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, a_t) + \lim_t \beta^t v(x_t) = w_p + \lim_t \beta^t v(x_t).$$

[36] Observe that the limit $\lim_T \sum_{t=0}^{T} \beta^t F(x_t, a_t)$ is well defined by the previous lemma.

As such, we only need to show that $\lim_t \beta^t v(x_t) = 0$.[37] Now,

[37] In other words, for all $\varepsilon > 0$ there exists an N_ε such that for all $n \geq N_\varepsilon$, $|\beta^t v(x_t)| < \varepsilon$.

$$|v(x_T)| = \frac{|v(x_T)|}{\phi(x_T)} \frac{\phi(x_T)}{\phi(x_{T-1})} \cdots \frac{\phi(x_1)}{\phi(x_0)} \phi(x_0).$$

Also,

$$\frac{|v(x_T)|}{\phi(x_T)} \leq \|v\|_\phi.$$

And, for all $t \geq 1$,

$$\begin{aligned} \phi(x_t) &\leq \max_{a \in \Gamma(x_{t-1})} \phi(r(a, x_{t-1}) \leq \frac{\theta}{\beta} \phi(x_{t-1}), \\ \to \frac{\phi(x_t)}{\phi(x_{t-1})} &\leq \frac{\theta}{\beta}. \end{aligned}$$

from this

$$\begin{aligned} |v(x_T)| &\leq \left(\frac{\theta}{\beta}\right)^{T-1} \|v\|_\phi \phi(x_0), \\ \to \beta^T |v(x_T)| &\leq \theta^{T-1} \beta \|v\|_\phi \phi(x_0). \end{aligned}$$

The right hand side goes to zero as $T \to \infty$. This shows that the value function is an upper bound to any feasible solution of the infinite horizon optimization problem, which shows that $v(x_0) \geq V(x_0)$. □

Given the fixed point of the Bellman operator, define recursively a path $g = \{(x_0, a_0), (x_1, a_1), \ldots\} \in \Pi(x_0)$ by,[38]

$$a_t \in \arg\max_{a \in \Gamma(x_t)} \{F(x_t, a) + \beta v(r(x_t, a))\}.$$

[38] The right hand side is a maximization problem of a continuous function $(F(x_t, a) + \beta v(r(x_t, a)))$ over a compact set $\Gamma(x_t)$ so the solution a_t and therefore the path g is well defined.

Lemma 6. *Let (X, A, Γ, F, β) be a regular problem and let v be the fixed point of the Bellman operator. Then for all $x_0 \in X$, $v(x_0) = w_g$.*

Proof. Let $g = \{(x_0, a_0), (x_1, a_1), \dots\}$ be defined as above. Then, we have,

$$\begin{aligned} v(x_0) &= F(x_0, a_0) + \beta v(x_1), \\ &= F(x_0, a_0) + \beta F(x_1, a_1) + \beta v(x_2), \\ &\dots, \\ &= \sum_{t=1}^{T} \beta^t F(x_t, a_t) + \beta^T v(x_T). \end{aligned}$$

Taking limits gives, $v(x_0) = w_g(x_0) + \lim_T \beta^T v(x_T)$ and we know that

$$\lim_T \beta^T v(x_T) = 0.$$

as was to be shown. □

Let us consider the following example,

$$\begin{aligned} \max_{c_0, c_1, \dots} \sum_{t=0}^{\infty} \beta^t \ln(c_t + 1) \text{ s.t. } k_{t+1} = k_t^\alpha - c_t, \\ \text{s.t. } 0 \le c_t \le k_t^\alpha, \\ k_0 \text{ given.} \end{aligned}$$

Here, capital k is the state variable and consumption c is the control variable, $\ln(c + 1)$ is the instantaneous utility function, $\beta \in (0, 1)$ is the discount factor and δ is the depreciation rate. In terms of the model outlined above, we have that $X = \mathbb{R}$, $A = \mathbb{R}$, $F = \ln(c_t + 1)$, $\Gamma(k) = \{c \in \mathbb{R}_+ : 0 \le c \le k^\alpha\}$ and $r(k, c) = k^\alpha - c$. We assume that $\alpha \le 1$.

The Bellman operator is given by,

$$(Tv)(k) = \max_{0 \le c \le k^\alpha} \ln(c + 1) + \beta v(k^\alpha - c).$$

In order for the fixed point of the Bellman operator to solve the problem, we need to find a function $\phi > 0$ a number M and $\theta < 1$ such that,

- $\max_{0 \le c \le k^\alpha} \ln(c + 1) \le M\phi(k)$.
- $\beta \max_{0 \le c \le k^\alpha} \phi(k^\alpha - c) \le \theta\phi(k)$.

Let's try the function $\phi(k) = \varepsilon \ln(k + 1) + 1 > 0$ which is strictly positive and continuous. We will determine the value of ε afterwards.

For the first condition,

$$
\begin{aligned}
\max_{0\le c\le k^\alpha} \ln(c+1) &= \ln(k^\alpha+1),\\
&\le \ln(k+1) + \frac{k^\alpha+1-(k+1)}{k+1},\\
&\le \ln(k+1) + \frac{(k+1)^\alpha}{k+1},\\
&= \ln(k+1) + (k+1)^{\alpha-1},\\
&\le \ln(k+1) + 1 \le M(\varepsilon\ln(k+1)+1)
\end{aligned}
$$

where M is such that $M\varepsilon > 1$. For the second condition,

$$
\begin{aligned}
\beta\left(\max_{0\le c\le k^\alpha} \varepsilon\ln(k^\alpha-c+1)+1\right) &= \beta\varepsilon\ln(k^\alpha+1)+\beta,\\
&\le \beta\varepsilon\left(\ln(k+1)+\frac{k^\alpha+1-(k+1)}{k+1}\right)+\beta,\\
&\le \beta\varepsilon(\ln(k+1)+1)+\beta,\\
&= \beta\varepsilon\ln(k+1)+\beta(1+\varepsilon),\\
&< \theta(\varepsilon\ln(k+1)+1)
\end{aligned}
$$

where $\beta(1+\varepsilon) < \theta$. So we know that in this case, the fixed point of the Bellman equation coincides with the optimal value of the infinite horizon optimization problem.

Let us now have a look at some of the properties of the fixed point of the Bellman operator. In this part, we will take the simplifying assumption that $A = X$ and $r(x,a) = a$. In other words, the optimization problem can be rewritten as,

$$v(x) = \max_{a\in\Gamma(x)} F(x,a) + \beta v(a).$$

Theorem 8. *Let (X,Γ,F,β) satisfy the assumptions of Definition 12. Let $F(.,a)$ be strictly increasing in each of its first arguments and assume that Γ is monotone in the sense that for $x \le x'$,*

$$\Gamma(x) \subseteq \Gamma(x').$$

Then, the fixed point v of the Bellman operator is strictly increasing.

Proof. Let $B'_\phi(X) \subseteq B_\phi(X)$ be the set of bounded (in the $\|.\|_\phi$ norm), continuous, weakly-increasing functions on X and let $B''_\phi(X) \subset B'_\phi(X)$ be the subset of strictly increasing functions. Since $B'_\phi(X)$ is a closed subset of $B_\phi(X)$ it suffices to show that $T[B'_\phi(X)] \subseteq B''_\phi(X)$.[39]

If $x' > x$ then by monotonicity of Γ: $\Gamma(x) \subseteq \Gamma(x')$. Let a' solve $\max_{a\in\Gamma(x)}\{F(x,a)+\beta v(a)\}$. Then,

$$
\begin{aligned}
(Tv)(x) = F(x,a') + \beta v(a')) &< F(x',a') + \beta v(a'),\\
&\le \max_{a\in\Gamma(x')} F(x',a) + \beta v(a) = (Tv)(x').
\end{aligned}
$$

[39] In other words, the Bellman operator maps weakly increasing functions into the set of strictly increasing functions.

□

Theorem 9. *Let (X, Γ, F, r, β) satisfy the assumptions of Definition 12. Let F be strictly concave, i.e. for all $\theta \in (0,1)$,*

$$F(\theta(x,a) + (1-\theta)(x',a')) \geq \theta F(x,a) + (1-\theta)F(x',a'),$$

with a strict inequality if $(x,a) \neq (x',a')$ and assume that Γ is convex in the sense that for all $\theta \in [0,1]$ and $x, x' \in X$,

$$a \in \Gamma(x), a' \in \Gamma(x') \to \theta a + (1-\theta)a' \in \Gamma(\theta x + (1-\theta)x').$$

Then v is strictly concave and $g(x) = \arg\max_{a \in \Gamma(x)} F(x,y) + \beta v(a)$ is a continuous, single-valued function.

Proof. Let $B'_\phi(X) \subseteq B_\phi(X)$ be the set of bounded continuous, weakly concave functions and let $B''_\phi(X) \subseteq B'_\phi(X)$ be the subset of strictly concave functions. It suffices to show that $T[B'_\phi(X)] \subseteq B''_\phi(X)$.
Let v be concave and let $x_0 \neq x_1$, $\theta \in (0,1)$ and set $x_\theta = \theta x_0 + (1-\theta)x_1$. Also let a_0 solve $\max_{a \in \Gamma(x_0)} F(x_0, a) + \beta v(a)$ and a_1 solve $\max_{a \in \Gamma(x_1)} F(x_1, a) + \beta v(a)$. Let $a_\theta = \theta a_0 + (1-\theta)a_1$. Then,

$$\begin{aligned}(Tv)(x_\theta) &\geq F(x_\theta, a_\theta) + \beta v(a_\theta), \\ &> \theta F(x_0, a_0) + (1-\theta)F(x_1, a_1) + \beta\theta v(a_0) + \beta(1-\theta)v(a_1), \\ &= \theta(Tv)(x_0) + (1-\theta)(Tv)(x_1).\end{aligned}$$

This shows that the Belman fixed point function is strictly concave. Given strict concavity,

$$\max_{a \in \Gamma(x)} F(x,a) + \beta v(a),$$

maximizes a strictly concave function. As such, the optimal value is unique so $g(x)$ is a function. As g is also u.h.c. (from Berge's maximum theorem) the function is continuous. □

Theorem 10. *Let (X, Γ, F, β) satisfy the assumptions of Definition 12 and let v be the fixed point of the Bellman operator. Let $F(x,a)$ be strictly concave in a, let $B'_\phi(X)$ be the set of bounded continuous, concave functions and let $v_0 \in B'_\phi(X)$. Let $\{(v_n, g_n)\}$ be defined as,*

$$\begin{aligned}v_{n+1} &= Tv_n, \\ g_n(x) &= \arg\max_{a \in \Gamma(x)} F(x,a) + \beta v_n(a).\end{aligned}$$

Then $g_n \to g$ pointwise. if X is compact, then $\|g_n - g\|_\phi \to 0$.

Proof. Let $B''_\phi(X)$ be the set of strictly concave bounded continuous functions. We know that for all n, $v_n \in B''_\phi(X)$. For $a \in \Gamma(x)$, let

$f_n(x,a) = F(x,a) + \beta v_n(a)$. We have that every function $f_n(x,y)$ is strictly concave. Also let $f(x,a) = F(x,a) + \beta v(a)$. Then,

$$\begin{aligned} |f_n(x,a) - f(x,a)| &= \beta |v_n(a) - v(a)|, \\ &= \phi(a)\beta \frac{|v_n(a) - v(a)|}{\phi(a)}, \\ &\leq \phi(x)\theta \|v_n - v\|_\phi \end{aligned}$$

This shows that $\|f_n(x,a) - f(x,a)\|_\phi \to 0$. As such $g_n(x) \to g(x)$ pointwise. If X is compact, we get that $\|g_n - g\|_\phi \to 0$. □

The following part provides assumptions for which the value function can be assumed to be differentiable. It uses the Benveniste Scheinkman theorem.

Theorem 11 (Benveniste and Scheinkman)**.** *Let $X \subseteq \mathbb{R}^l$ be a convex set, let $V : X \to \mathbb{R}$ be concave, let x_0 be in the interior of X and let D be a neighbourhood of x_0. If there is a concave, differentiable function $W : X \to \mathbb{R}$ with $W(x_0) = V(x_0)$ and $W(x) \leq V(x)$ for all $x \in D$ then V is differentiable at x_0 and,*

$$V_i(x_0) = W_i(x_0).$$

Proof. Any subgradient p of $V(x_0)$ must satisfy for all $x \in D$,

$$W(x) - W(x_0) \leq V(x) - V(x_0) \leq p(x - x_0).$$

Since W is differentiable, the vector p is unique, so V has a unique subgradient, so V must be differentiable. □

Theorem 12. *Let (X, Γ, F, β) satisfy assumptions of theorem 12 and assume that F is strictly concave and Γ is convex. Assume that F is C^1. Let v be the fixed point of the Bellman operator and let g be the unique optimal value function. If x_0 is in the interior of X and $g(x_0)$ is in the interior of $\Gamma(x_0)$, then v is C^1 at x_0 and*

$$v_i(x_0) = F_i[x_0, g(x_0)].$$

Proof. As F is strictly concave and Γ is convex, g is a function. Also, since $g(x_0)$ is in the interior of $\Gamma(x_0)$ and Γ is continuous, $g(x_0)$ is in the interior of $\Gamma(x)$ for all x in a neighborhood D of x_0. Define W on D by

$$W(x) = F(x, g(x_0)) + \beta v(g(x_0)).$$

Since F is concave and differentiable, this function is concave and differentiable in x. Also,

$$W(x) \leq \max_{a \in \Gamma(x)} F(x,a) + \beta v(a) = v(x).$$

with equality at x_0. The result follows from Benveniste and Sheinkman theorem. □

Euler equations

There is a second more classical mode of attack on the dynamic optimization problem. This alternative approach is based on first order conditions. As before, consider the following infinite horizon optimization problem.

$$\max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ s.t. } x_{t+1} \in \Gamma(x_t).$$

Consider that this problem can be solved and take an optimal solution $\{x_t^*\}_{t=0}^{\infty}$. Now, consider the following problem,

$$\max_{x_{t+1}} F(x_t^*, x_{t+1}) + \beta F(x_{t+1}, x_{t+2}^*) \text{ s.t. } x_{t+1} \in \Gamma(x_t^*), x_{t+2}^* = \Gamma(x_{t+1}).$$

Given the optimality of x_t^* and x_{t+2}^*, we need that x_{t+1}^* is the optimal solution to this problem. Now, assuming that F is differentiable and if x_{t+1}^*, x_{t+2}^* are in the interior of $\Gamma(x_t^*)$ and $\Gamma(x_{t+1}^*)$ we obtain the following first order condition,

$$F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*) = 0.$$

This equation is known as the Euler equation. If the solution is interior, then this condition is necessary for optimality. Usually the set of Euler equations is completed by adding a so called **transversality condition** namely,

$$\lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}) x_T^* = 0.$$

It can be shown that if the Euler equations are satisfied, $F_x > 0$ and the transversality hold, then if F is a concave function, it must be that the solution $\{x_1^*, \ldots, x_n^*, \ldots\}$ is indeed optimal. To see this, let $\{x_1, x_2, \ldots\}$ be another feasible path. If F is concave, then,

$$\sum_{t=0}^{T} \beta^t (F(x_t, x_{t+1}) - F(x_t^*, x_{t+1}^*) \leq \sum_{t=0}^{T} \beta^t F_x(x_t^*, x_{t+1}^*) \cdot (x_t - x_t^*) + \sum_{t=0}^{T} \beta^t F_y(x_t^*, x_{t+1}^*) \cdot (x_{t+1} - x_{t+1}^*).$$

rearranging terms gives,

$$\begin{aligned}\sum_{t=0}^{T} \beta^t (F(x_t, a_t) - F(x_t^*, a_t^*)) \leq & \sum_{t=0}^{T-1} \beta^{t-1} \left[\beta F_x(x_{t+1}^*, x_{t+2}^*) + F_y(x_t^*, x_{t+1}^*)\right] \cdot (x_{t+1} - x_{t+1}^*), \\ & + \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1} - x_{T+1}^*).\end{aligned}$$

The summations in the first line are equal to zero. Then,

$$\begin{aligned}\sum_{t=0}^{T} \beta^t (F(x_t, a_t) - F(x_t^*, a_t^*)) &\leq \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1} - x_{T+1}^*), \\ &= -\beta^{T+1} F_x(x_{t+1}^*, x_{t+2}^*)(x_{T+1} - x_{T+1}^*), \\ &\leq \beta^{T+1} F_x(x_{t+1}^*, x_{t+2}^*) x_{T+1}^*.\end{aligned}$$

using $F_x \geq 0$ and $x_T \geq 0$ for all T. Now (taking $T \to \infty$) the solution is optimal whenever,

$$\lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1})x_T^* = 0.$$

which is indeed the case if the transversality condition holds.

As an example, consider the Bellman equation of the optimal growth problem,

$$v(k) = \max_{k' \leq Ak^\alpha} \ln(Ak^\alpha - k') + \beta v(k').$$

The first order condition gives,

$$\frac{1}{k' - Ak^\alpha} + \beta v'(k).$$

Then from the envelope theorem we get,

$$v'(k) = \frac{A\alpha k^{\alpha-1}}{k' - Ak^\alpha}$$

Updating one period and substitution gives the Euler equation,

$$\frac{1}{k_{t+1} - Ak_t^\alpha} + \beta \frac{\alpha A k_{t+1}^{\alpha-1}}{k_{t+2} - Ak_{t+1}^\alpha} = 0.$$

Which is a second order difference equation. The transversality condition requires that,

$$\lim_{t\to\infty} \frac{A\alpha\beta^t k_t^\alpha}{k_{t+1} - Ak_t^\alpha} = 0.$$

Numerical methods

In the previous chapter we say that the solution of the infinite horizon dynamic programming problem could be restated in terms of a solution ot the Bellman equation. In many cases there are no closed form solutions to this Bellman equation[40] On the other hand, we have also seen that the unique solution of the Bellman equation can be found as the fixed point of a contraction mapping. Additionally, this fixed point can be approximated infinitely close by iterating the contraction mapping operator. In this sense, it is possible to approximate our solution via simulation methods. These methods are of course finite in nature and provide therefore only an approximation to the true fixed point. The simplest simulation technique is the value function iteration.

[40] If there are solutions, they are mainly used for pedagogical purposes and are only available in a few special settings.

Value function iteration

The contraction mapping theorem tells us that the solution of the Bellman equation can be found by iterating the Bellman operator T

$$(Tv)(x) = \max_{a\in\Gamma(x)} \{F(x,a) + \beta v(r(x,a))\}.$$

When X is an infinite set, it is impossible to compute (Tv) for every value $x \in X$. To remedy this, one usually approximates the state space X by a finite set, using a grid for the possible values of x,

$$X = \{x_1, x_2, \ldots, x_n\}.$$

Also, the correspondence $\Gamma : X \to A$ is now replaced by a non-empty correspondence from the finite set X to a finite set A. The instantaneous return function $F : X \times A \to \mathbb{R}$ takes only takes on a finite set $X \times A$, so it is bounded by definition. It can be shown that when restricted to this finite set X, the Bellman operator still is a contraction mapping.[41] Finding, or approximating, the fixed point of T takes the following steps:

[41] Observe that F is bounded, so the conditions of Definition 12 are satisfied with $\phi(x) = 1$ which means that the Bellman operator $T : B(X) \to B(X)$ where $B(X)$ is the set of bounded functions on X has a unique fixed point by Blackwell's theorem. Also, the corresponding policy correspondence,

$$G(x) = \arg\max_{a\in\Gamma(x)} F(x,a) + v(r(x,a)),$$

is non-empty.

1. Decide on a (fine enough) grid for the state space X and control space A.[42]

2. Decide on some tolerance level ε.[43]

3. Decide on an initial bounded function $v_0 : X \to \mathbb{R}$. Initiate the iteration round $t = 0$.

4. (a) Compute for all $x \in X$: $v_{t+1}(x) = \max_{a \in \Gamma(x)} \{F(x,a) + \beta v_t(r(x,a))\}$.
   (b) Save the policy function $G(x) = \arg\max_{a \in \Gamma(x)} \{F(x,a) + \beta v_t(r(x,a))\}$.
   (c) Repeat as long as $\|v_{t+1} - v_t\| > \varepsilon$, each time iterating the counter $t = t + 1$.

5. The final function v_t and $G(x)$ should be a good approximation to the fixed point of the Bellman operator and the corresponding policy function.

[42] Often the problem can be reformulated such that the control space and state space coincide.

[43] This should be a sufficiently small number.

In order to get a better grasp of the algorithm, let us work out an example. Consider a representative consumer with CRRA utility function,[44]

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}.$$

[44] CRRA stands for constant relative risk aversion. The relative risk aversion of the utility function $u(.)$ is given by,

$$-\frac{u''(c)}{u'(c)} c.$$

The amount of capital in period $t + 1$ is determined using the following law of motion,

$$k_{t+1} = k_t^\alpha - c + (1 - \delta) k_t.$$

Here k_t^α ($\alpha \in [0,1]$) gives the output produces by the capital stock k_t and δ is the depreciation rate. Given that $c \geq 0$, we can substitute the law of motion into the objective function to get the following Bellman operator,

$$(Tv)(k) = \max_{k' \leq k^\alpha + (1-\delta)k} \frac{(k^\alpha + (1-\delta)k - k')^{1-\sigma} - 1}{1 - \sigma} + \beta v(k').$$

We make the following choice of parameters.[45]

[45] We use Matlab to simulate the model.

```
%parameter values
par.sigma = 1.5;
par.delta = 0.1;
par.beta = 0.95;
par.alpha = 0.3;
```

Next, we have to decide on a grid of feasible values of k,

$$K = \{k_1, \ldots, k_N\},$$

Let's choose 1000 grid points in a neighbourhood around the steady state. Each point in the grid is chosen a distance Δ apart. So if k_{min} is the smallest grid point then the n-th grid point will have the value

$$k_n = k_{\text{min}} + (n-1)\Delta.$$

We also need an initial value function $v_0(.)$ and an initial policy function $g(.)$. The value function $v_0(.)$ is simply encoded as a vector of N numbers that relate a certain value to each value k_n in the grid. The policy function g is also determined by an N dimensional vector. The nth value of g gives a number between 1 and N that points to the optimal choice of k'. For example if $g_i = j$ this means that the optimal choice when the state is k_i is given by k_j, i.e. $g(k_i) = k_j$.

```
par.ngrid = 1000; %grid size

%steady state kapital stock
kast = ((1 - par.beta*(1-par.delta))/(par.alpha*par.beta))^(1/(par.alpha-1));

%constructing the grid
dev          = 0.9; %deviation from steady state
kmin         = (1-dev)*kast;
kmax         = (1+dev)*kast;
par.Delta    = (kmax - kmin)/(par.ngrid-1);
K            = linspace(kmin,kmax,par.ngrid)'; %levels of possible capital, X
v            = zeros(par.ngrid,1); %initial value
Tv           = zeros(par.ngrid,1); %next value
g            = zeros(par.ngrid,1); %policy function
```

Next, we go to the main loop for the program. We choose a tolerance level of $\varepsilon = 10^{-6}$. As long as the distance

$$\|Tv - v\| = \max_{k\in K} |Tv(k) - v(k)|,$$

is larger than ε, the program updates the value of v to Tv using a call to the function $fviter$. This function provides the value of $(Tv)(k)$ and $g(k)$. Once, the algorithm is finished, the values of g are assigned and also the consumption and utility levels corresponding to the various initial capital levels are computed.

```
crit = 1; %initial critical value
par.eps = 1e-6; %tolerance level

iter = 0;
while crit > par.eps %while convergence is not reached

    %bellman iteration
    [Tv, g] = fviter(par, v, K);

    %compute ||Tv - v||
    crit = max(abs(Tv-v)); %compute convergence criterion

```

```
        %update value function
        v = Tv;

    end

    % final solution
    K_next   = K(g);
    C        = K.^par.alpha+ (1-par.delta)*K-K_next;
    U        = (C.^(1-par.sigma)-1)/(1-par.sigma);
```

The crucial part of the algorithm is the function $fviter$. This function loops over all possible values k_i in the grid in order to retrieve the value of $(Tv)(k_i)$. Towards this end, we first needs to retrieve the set $\Gamma(k_i)$. We know that for all $k_n \in \Gamma(k_i)$,

$$\begin{aligned} & k_n \le k_i^\alpha + (1-\delta)k, \\ \leftrightarrow & k_{\min} + (n-1)\Delta \le k_i^\alpha + (1-\delta)k_i, \\ \leftrightarrow & n \le \frac{k_i^\alpha + (1-\delta)k_i - k_{\min}}{\Delta} + 1. \end{aligned}$$

So we take the largest integer below $\frac{k_i^\alpha+(1-\delta)k_i-k_{\min}}{\Delta} + 1$ as our largest possible index for the capital stock. The final part calculates for each possible value in $\Gamma(k_i)$ the consumption and utility value. The last line of code chooses $(Tv)(k_i)$ as the maximum of these values.

This code and the following are not the most efficient implementations. They are more designed so you understand the method without adding coding complications.

```
    function [Tv, g] = fviter(par, v, K)

    Tv       = zeros(par.ngrid,1);
    kmin     = K(1);
    g        = zeros(par.ngrid,1);

    for i = 1:par.ngrid
        k = K(i); %captial level

        % compute Gamma(k)
        tmp          = (k^par.alpha + (1-par.delta)*k - kmin);
        gamma_max    = min(floor(tmp/par.Delta) + 1, par.ngrid);

        % consumption and utility
        C    = k^par.alpha + (1-par.delta)*k-K(1:gamma_max);
        U    = (C.^(1-par.sigma)-1)/(1-par.sigma);

        % find value function
        [Tv(i), g(i)] = max(U+par.beta*v(1:gamma_max));
    end
```

When running this code, we find convergence after 194 iterations in approximately 26.5 seconds. This is still a very simple example. More elaborate models with possible multidimensional grids take much more time.

The convergence rate of the Bellman operator has a rate of β. For many economic models, it is natural to choose β close to 1. Convergence of value function iteration method is particularly slow if β is chosen to be close to one.

Interpolation

A FIRST POSSIBLE improvement on the speed of the algorithm is to have a smaller grid on the capital stock. But try to have a reasonable good estimation of the Bellman operator.

$$(Tv)(k) = \max_{a \in \Gamma(x)} \{F(x,a) + \beta v(a)\}.$$

The idea behind interpolation is choose a fine grid for the values of a and to determine the value of $v(a)$ by interpolation on the values of v on this small grid for a.

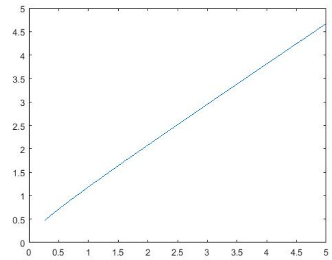

Figure 1: Present capital stock versus next periods capital stock.

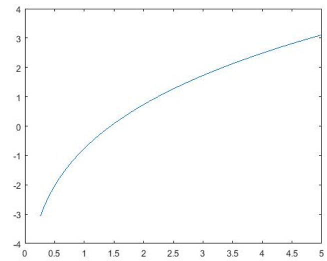

Figure 2: Value function.

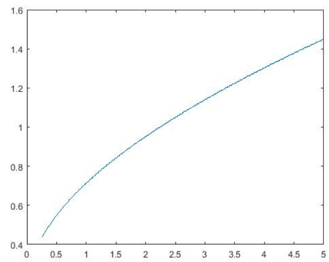

Figure 3: Consumption as a function of capital.

1. Decide on the a small grid for the state space X.
2. Decide on a dense grid for the choice variable a.
3. Pick an initial bounded function $v_0 : X \to \mathbb{R}$. Initiate the iteration round to one, $t = 1$

   (a) Compute for all $x \in X$: $v_{t+1}(x) = \max_{a \in \Gamma(x)}\{F(x,a) + \beta \tilde{v}_t(a)\}$, where $\tilde{v}_t(a)$ is computed by interpolation of the function v_t : $A \to \mathbb{R}$.

   (b) Store the policy function

   $$G(x) = \arg\max_{a \in \Gamma(x)} \{F(x,a) + \beta v_t(a)\}.$$

   Iterate these two steps for as long as $\max_{x \in X} |v_t(x) - v(x)| > \varepsilon$,
4. The final function v_t and $G(x)$ should be a good approximation to the fixed point of the Bellman operator.

The code for this adaptation is given below. We have chosen splines to interpolate.

```
%parameter values
par.sigma   = 1.5;
par.delta   = 0.1;
par.beta    = 0.95;
par.alpha   = 0.3;

%parameters for iteration
par.ngrid = 20; %grid size
par.ygrid = 1000;

kast = ((1 - par.beta*(1-par.delta))/(par.alpha*par.beta))^(1/(par.alpha-1)); %steady state kapital st

%making of the grid
```

```
dev         = 0.9; %deviation from steady state
kmin        = (1-dev)*kast;
kmax        = (1+dev)*kast;
par.Delta   = (kmax - kmin)/(par.ngrid-1);
K           = linspace(kmin,kmax,par.ngrid)'; %levels of possible capital

ymin    = 0;
ymax    = kmax^par.alpha + (1-par.delta)*kmax;
Y       = linspace(ymin,ymax,par.ygrid)';

v   = zeros(par.ngrid,1); %initial value
Tv  = zeros(par.ngrid,1); %next value
g   = zeros(par.ngrid,1); %policy function

crit        = 1; %initial critical value
par.eps     = 1e-6; %tolerance level

iter = 0;
while crit > par.eps %while convergence is not reached

    for i = 1:par.ngrid
        k       = K(i);
        knext   = Y;
        c       = k^par.alpha +(1- par.delta)*k - knext;
        A       = c>0; %find positive consumption values
        util    = (c(A,:).^(1-par.sigma)-1)/(1-par.sigma);
        vy      = interp1(K,v,knext(A,:),'spline');

        [Tv(i),g(i)] = max(util+ par.beta*vy);
    end

    crit = max(abs(Tv-v)); %compute convergence criterion
    v = Tv;
    iter = iter+1;

end
%
% final solution
%
K_next   = Y(g);
C        = K.^par.alpha+ (1-par.delta)*K-K_next;
U        = (C.^(1-par.sigma)-1)/(1-par.sigma);
```

The algorithm ends after 200 iterations and in 1.18 seconds. Much faster. The increase in speed is maily due to the large decrease of the grid size. Unfortunately, the value function is now only known at a smaller number of points and the interpolation function might be a bad guess of the true value function. It is also not possible to prove convergence of the algorithm.

Howard improvement

THE SPEED OF THE function iteration algorithm is mainly determined by the optimization routine. As such, computational improvements should be aimed at reducing the number of optimization iterations.

This is the idea behind the Howard improvement algorithm. Let H be the set of potential policy functions $H = \{h : X \to A : h(x) \in \Gamma(x)\}$. For any h define an operator T_h such that,

$$(T_h v)(x) = F(x, h(x)) + \beta v(h(x)).$$

It is easily verified that the mapping $T_h : B(X) \to B(X)$ satisfies the conditions of Blackwell's theorem so it has a fixed point which can be obtained by iteration. This fixed point satisfies $v(x) = F(x, h(x)) + \beta v(h(x))$ so it computes the value of the infinite horizon problem under the constraint that the policy function h is used in every period. Importantly, the computation of this fixed point does not require any optimization routine.

The Howard improvement procedure takes the following form.

1. Decide on a grid X.
2. Pick any value function v_0.
3. initiate the loop at $t = 1$ for all t, do the following
   (a) Find an updated policy function h_t such that,
   $$h_t(x) = \arg\max_{a \in \Gamma(x)} F(x, a) + \beta v_{t-1}(a).$$
   (b) find v_t as the unique fixed point of $T_{h_{t+1}}$, i.e.
   $$v_t(x) = F(x, h_t(x)) + \beta v_t(h_t(x)).$$
   (c) iterate steps (a) and (b) until $\|v_t - v_{t-1}\| < \epsilon$.

Howard's improvement algorithm first converge on the value function given a policy function h_t. Once this function is found, the policy function h_t is updated. The advantage of this algorithm is that it requires fewer optimization iterations. Given that this is the most costly step, the algorithm is usually faster.

Lemma 7. *The sequence of functions v_n converges to to the fixed point of the Bellman operator T.*

Proof. We will show that $v_0 \leq Tv_0 \leq v_1 \leq Tv_1 \leq \ldots$. This is an increasing sequence in a bounded set, so this converges to the value $\sup_t v_t$ which is a fixed point of the Bellman operator T.

First we show that for all n, $(Tv_n) \geq v_n$. Indeed,

$$\begin{aligned}(Tv_n)(x) &= \max_{a \in \Gamma(x)} F(x, a) + \beta v_n(a), \\ &\geq F(x, h_n(x)) + \beta v_n(h_n(x)) = v_n(x).\end{aligned}$$

The last equality follows from the fact that v_n is a fixed point of the operator T_{h_n}. Additionally, $(Tv_n) = (T_{h_{n+1}}v_n)$. Indeed,

$$(Tv_n)(x) = F(x, h_{n+1}(x)) + \beta v_n(h_{n+1}(x)) = (T_{h_{n+1}}v_n)(x).$$

Then,

$$\begin{aligned}(T^2_{h_{n+1}}v_n)(x) - (T_{h_{n+1}}v_n)(x) =& F(x, h_{n+1}(x)) + \beta(T_{h_{n+1}}v_n)(h_{n+1}(x)),\\ &- F(x, h_{n+1}(x)) - \beta v_n(h_{n+1}(x)),\\ &= \beta\left[(Tv_n)(h_{n+1}(x)) - v_n(h_{n+1}(x))\right] \geq 0.\end{aligned}$$

As such, $(T^2_{h_{n+1}}v_n) \geq T_{h_{n+1}}v_n = Tv_n$.

Now let us show that $v_{n+1} \geq Tv_n$. In order to do this, we show that $(T^m_{h_{n+1}}v_n) \geq (T^{m-1}_{h_{n+1}}v_n)$ for all $m \geq 2$. For $m = 2$, the proof is given above. Now for the induction step,

$$\begin{aligned}&(T^m_{h_n}v_{n-1})(x) \geq (T^{m-1}_{h_n}v_{n-1})(x),\\ \leftrightarrow& F(x, h_n(x)) + \beta(T^{m-1}_{h_n}v_{n-1})(h_n(x)) \geq F(x, h_n(x)) + \beta(T^{m-2}_{h_n}v_{n-1})(h_n(x)),\\ \leftrightarrow&(T^{m-1}_{h_n}v_{n-1})(h_n(x)) \geq (T^{m-2}_{h_m}v_{n-1})(h_n(x)).\end{aligned}$$

which is indeed true by the induction hypothesis. Given that $v_n = \lim_m(T^m_h v_{n-1})$, we have that

$$v_{n+1} \geq (T^2_{h_{n+1}}v_n) \geq Tv_n,$$

as was to be shown. □

The Howard algorithm can be implemented using the following change to the loop of the program. First, the function $fviter$ is called to update the policy function g. Then the function pfiter is called to find the fixed point of the policy evaluation function.

```
while crit > par.eps %while convergence is not reached

    %Bellman iteration
    [Tv, g]= fviter(param, v, K);

    %policy function iteration
    v = pfiter(param,Tv, K, g);

    %compute \|v - Tv\|
    crit = max(abs(v- Tv)); %compute convergence criterion
end
```

The function $pfiter$ looks like this.

```
function v = pfiter(params,v, K, g)

while crit > par.eps
```

```

    %consumption and utility
    C = K.^alpha + (1-delta)*K - K(g);
    U = (C.^(1-sigma)-1)/(1-sigma);

    %iteration
    Tv = U + beta*v(g);

    %compute ||Tv - v||
    crit = max(abs(Tv-v));
    v = Tv;
end
```

Convergence is reached in 16 iterations of the function *fviter*, in 2.61 seconds. Instead of using the function $pfiter$ it is sometimes possible to explicitly solve this step. Observe that the fixed point of the operator T_h satisfies,

$$v(k) = F(k, h(k)) + \beta v(h(k)).$$

This can be written in vector notation as,

$$v = F(K, K(g)) + \beta Q v.$$

where Q is an $N \times N$ matrix with a 1 at position i, j if and only if $g_{i,j} = 1$. This system can be solved for v,

$$v = (I - \beta Q)^{-1} F(K, K(g)).$$

This gives the following alternative to $pfiter$.

```
function v = howard_inv(par,Tv,K,g)

K_next = K(g);

%consumption and utility
C = K.^par.alpha + (1-par.delta)*K-K_next;
U = (C.^(1-par.sigma)-1)/(1-par.sigma);

%construct matrix Q
Q = sparse(par.ngrid, par.ngrid);
for i = 1:par.ngrid
    Q(i,g(i)) = 1;
end

%v = (I - beta Q)^{-1} U
v = (speye(par.ngrid)-par.beta*Q)\U;
```

Convergence is reached in 16 iterations, now in 2.31 seconds.

Parametric interpolation

THE SPEED OF THE iteration is slowed down by the size of the grid. The denser the grid, the better the approximation but also the longer the computation time. This is especially true when analyzing problem where the state space is more dimensional. If we have a grid of 100 points in a one dimensional setting to reach a certain level of accuracy, it takes approximately 100^2 grid points in a two dimensional problem, 100^3 grid points in a three dimensional problem and so on. As such, the problem becomes untractable even if there are only a moderate number of dimensions.

A possible way to avoid this problem is to exploiting the smoothness of typical economic examples to approximate the value function by some flexible functional form.[46] This flexible functional form usually depends on some some parameter values. The idea is then to iterate on the value of these parameters to approximate the value function as close as possible. The algorithm then takes the following steps,

[46] Notice however, that the convergence of the function iteration algorithm with nonlinear approximations of the value function is not guaranteed.

1. Decide on some flexible functional form $v(x, \sigma)$ and a grid of possible values X. Here σ is a set of parameters that fully determines the function $v(., \sigma)$.

2. Choose a critical value ε and a starting value for the parameters σ_0. Set $t = 1$,

3. Compute,
$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y, \sigma_{t-1}).$$

4. Update the vector of parameters σ_t to make $v(x, \sigma_t)$ as close as possible to $(Tv)(x)$.

5. if $\|v(x, \sigma_t) - v(x, \sigma_{t-1})\| < \varepsilon$ then stop. Else set $t = t + 1$, and go back to step 2.

For this algorithm to be implementable we need to choose a specific functional form $v(., \sigma)$ and we need to decide on the updateing rule in step 4. For the first issue, one usually choses either a combination of polynomials, neural networks or splines. Concerning step 4, one can uses the value of σ_{t+1} that minimizes the sum of squares.

$$\sigma_{t+1} = \arg\min_{\sigma} \sum_{x \in X} \left[(Tv)(x) - v(x, \sigma)\right]^2.$$

Fortunately, the algorithm is usually much faster in high dimensional settings, with a comparable accuracy.[47]

[47] For the least squares estimates to be of high quality, the number of grid points should be considerably larger than the number of parameters θ to be estimated.

Let us have a look at an example where $v(.)$ is approximated using polynomials. One attractive option is the use of Chebychev polynomials. The n-th Chebychev polynomial is defined on the interval $[-1,1]$ and has the form,

$$T_n(x) = \cos(n\cos^{-1}x).$$

Or equivalently, $T_n(\cos(\theta)) = \cos(n\theta)$. We know that,

$$\begin{aligned}
T_0(\cos(0)) &= \cos(0\theta) = 1,\\
T_1(\cos(\theta)) &= \cos(\theta),\\
T_2(\cos(\theta)) &= \cos(2\theta) = 2\cos^2\theta - 1,\\
T_3(\cos(\theta)) &= \cos(3\theta) = 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

This gives for the first terms,

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x.
\end{aligned}$$

Now,

We have that $e^{iz} = (\cos z + i\sin z)$. The second line is obtained by gathering the real terms of the two expansions.

$$\begin{aligned}
e^{i(n+1)\theta)} &= e^{in\theta}e^{i\theta},\\
\to \cos((n-1)\theta) &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta).
\end{aligned}$$

We would like to get rid of the sin functions. We can do this by considering the following expansion,

$$\begin{aligned}
e^{i(n-1)\theta} &= e^{in\theta}e^{-i\theta},\\
\to \cos((n-1)\theta) &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta).
\end{aligned}$$

Adding the two together gives,

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)\cos(\theta).$$

As such,

$$\begin{aligned}
T_{n+1}(x) + T_{n-1}(x) &= 2T_n(x)x,\\
\to T_{n+1}(x) &= 2T_n(x)x - T_{n-1}(x).
\end{aligned}$$

From this, we can compute,

$$\begin{aligned}
&T_4(x) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1,\\
&T_5(x) = 16x^5 - 16x^3 + 2x - 4x^3 + 3x = 16x^5 - 20x^3 + 5x,\\
&\ldots
\end{aligned}$$

This shows that they can easily be computed iteratively. The roots of the polynomial T_n are equal to the values $\cos\left(\frac{(2k+1)\pi}{2n}\right)$. Indeed, we must have that,

$$
\begin{aligned}
T_n(\cos(\theta)) &= \cos(n\theta) = 0, \\
\rightarrow n\theta &= \frac{\pi}{2} + k\pi, \\
\rightarrow \theta &= \frac{\pi(2k+1)}{2n}, \\
\rightarrow x &= \cos\left(\frac{\pi(2k+1)}{2n}\right).
\end{aligned}
$$

The grid points are often chosen to be these zeros.

The value function $v(x;\sigma)$ can be chosen such that,

$$
v(x;\sigma) = \sum_{t=1}^{N} \sigma_t T_n\left(2\frac{x-\underline{x}}{\overline{x}-\underline{x}} - 1\right).
$$

This is a linear combination of Chebychev polynomials. The normalization $2\frac{(x-\underline{x})}{\overline{x}-\underline{x}} - 1$ is done to rescale the state variables to the interval $[-1,1]$.

To start we begin by some parameter initializations. The parameters are the same as before. On line 30, we define the Chebychev polynomials and we initiate estimates for our parameters by regressing these on the estimate of our value function.[48]

[48] In matlab the OLS estimates are quickly computes using the command $X\backslash v$.

```
1  %parameter values
2  par.sigma   = 1.50;
3  par.delta   = 0.10;
4  par.beta    = 0.95;
5  par.alpha   = 0.30;
6
7  par.ngrids  = 20;       %size of the grid
8  par.n       = 10;       %number of Chebychev polynomials
9  par.epsi    = 1e-6;     %small value
10
11
12 crit    = 1;        %initiation critical value
13 iter    = 1;        %counter for number of iterations
14
15
16 %steady state
17 ks = ((1-par.beta*(1-par.delta))/(par.alpha*par.beta))^(1/(par.alpha-1));
18 dev = 0.9;
19
20 par.kmin = (1-dev)*ks;      %minimal capital stock
21 par.kmax = (1+dev)*ks;      %maximal capital stock
22
23 rk = cos((2*[1:par.ngrids]'-1)*pi/(2*par.ngrids));   %grid for Chebychev polynomials
24 K = par.kmin+(rk+1)*(par.kmax-par.kmin)/2;      %grid for capital stock values
25
26 %starting value
27 v = ((K.^par.alpha).^(1-par.sigma)-1)/((1-par.sigma)*(1-par.beta));
28
29 %chebychev functions
30 X = chebychev(rk,par.n);
```

```
%starting value for coefficients
sig = X\v;


Tv = zeros(par.ngrids,1);
Knext = K;
```

The function $chebychev(rk, n)$ creates the values of the n Chebychev polynomials on a grid given by rk. In order to compute theis, we consider some cases where $n = 0$ or 1. If n is larger or equal to two, we define the polynomials iterative,

$$T_n(x) = 2xT_{n-1}(x) - T_{n_2}(x).$$

```
function Tx = chebychev(x,n)

X = x(:);
lx = size(X,1);
if n<0;
error('n should be a positive integer');
end

switch n;
case 0;
    Tx = [ones(lx,1)];
case 1;
    Tx = [ones(lx,1) X];
otherwise
    Tx = [ones(lx,1) X];
    for i= 3:n+1
        Tx = [Tx 2*X.*Tx(:,i-1)-Tx(:,i-2)];
    end
end
```

Let's go back to the main program. The next part is the main loop. On line 11 we are minimizing a function tv which computes the function value

$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y,\sigma).$$

Then the estimates of σ are updated and the criterium $\|Tv - v\|$ is computed to determine when to stop the algorithm. The function $fmincon$ is a minimization routine, so we have to minimize $-F(x,y) - \beta v(y,\sigma)$. This is why we change signs on line 12.

```
1 options = optimset('Display','off');
2 while crit > par.epsi;
3     k0 = K(1);
4     for i = 1:par.ngrids
5         %captial stock
6         k = K(i);
7         par.k = k;
```

```
        %upper bound
        b = k^par.alpha + (1-par.delta)*k;
        %optimal value of next period capital stock
        [Knext(i), Tv(i)] = fmincon(@(x)tv(x,par, sig),Knext(i), [], [],[],[],0,b,[], options);
        Tv(i) = -Tv(i);
    end
    %update estimators
    sig = X\Tv;

    crit = max(abs(Tv-v));
    disp(crit);

    v = Tv;
    iter = iter+1;
end
```

The function to be minimized is given by *tv* and is given by the following code. After retrieving some parameter values, it computes the value function $v(y, \sigma)$ on line 13 via a separate function *value*, the consumption and utility and returns the value of $res = -F(x, y) - \beta v(y, \sigma)$.

```
1     function res = tv(kp, par, theta)
2
3     alpha   = par.alpha;
4     beta    = par.beta;
5     delta   = par.delta;
6     sigma   = par.sigma;
7     kmin    = par.kmin;
8     kmax    = par.kmax;
9     n       = par.n;
10    k       = par.k;
11
12    %value next period
13    v = value(kp,[kmin kmax n], theta);
14    %consumption
15    c = k.^alpha + (1-delta)*k - kp;
16    %utility
17    util = (c.^(1-sigma)-1)/(1-sigma);
18    %objective function to be minmized
19    res = -(util+beta*v);
```

The final program is *value* which computes $v(x, \sigma)$. It does so by rescaling the capital stocks and computing,

$$v(x, \sigma) = \sum_{i=1}^{N} \sigma_n T_n(w).$$

where w are the rescaled capital stock values.

```
function v = value(k,param,theta)

kmin = param(1);
kmax = param(2);
n = param(3);
k = 2*(k - kmin)/(kmax-kmin)-1;
```

```
v = chebychev(k,n)*theta;
```

The program terminates after 243 iterations in 32.30 seconds.[49] The final parameter values are given by,

parameter	value
σ_0	0.8237
σ_1	2.7804
σ_2	-0.6601
σ_3	0.2370
σ_4	-0.1028
σ_5	0.0515
σ_6	-0.0260
σ_7	0.0113
σ_8	-0.0062
σ_9	0.0050
σ_{10}	-0.0028

[49] Here, we see that the computation time is much longer than before. However, the advantage is that we can approximate the value function for any particular value of k. Judd and Solnick (1994) successfully applied this technique to the optimal growth model and found that the approximations was very good and dominates to a lot of other methods.

Some applications

LET US HAVE a look at some applications of dynamic programming under certainty.

Optimal tree growth

CONSIDER A TREE whose growth is described by a function $h : \mathbb{R}_+ \to \mathbb{R}_+$.[50] Assume that the price of wood is p per unit of height, and the interest rate r are both constant over time. For simplicity assume that $p = 1$ and $\beta = 1/(1+r)$. It is costless to cut down the tree.

[50] If k_t is the length of the tree in period t then $k_{t+1} = h(k_t)$ is the height of the tree tomorrow.

If the tree cannot be replanted, the problem in each period is either to cut the tree or not. If the tree is cut in period t then the value is given by $v(k_t) = k_t p = k_t$ and there is no value thereafter. If the tree is not cut in period t then the value is given by $v(k_t) = \beta v(h(k_t))$. Each period, the problem is either to cut the tree or not. As such,

$$v(k_t) = \max_{c=\{0,1\}} \{k_t c + (1-c)\beta v(h(k_t))\}.$$

Here c is a binary variable that decides whether to cut the tree or not. Observe that his problem can be rewritten as,

$$v(k_t) = \max\{k_t; \beta v(h(k_t))\}.$$

The first choice is taken when the tree is cut while if the second option is take the tree is not cut. Assume that there is a maximum height that the tree can take, $k \in [0, H]$.

Lemma 8. *The operator $(Tv)(k) = \max\{k, \beta v(h(k))\}$ is a contraction mapping from the set of bounded functions $B([0, H])$ to $B([0, H])$.*

Proof. We check Blackwell's theorem. If $v \leq w$ then

$$(Tv)(k) = \max\{k, \beta v(h(k))\} \leq \max\{k, \beta w(h(k))\} = (Tw)(k),$$

which shows monotonicity. For additivity,

$$\begin{aligned}(Tv+a)(k) &= \max\{k, \beta(v+a)(h(k))\},\\ &= \max\{k, \beta v(h(k)) + \beta a\},\\ &\leq \max\{k+\beta a, \beta v(h(k)) + \beta a\} = (Tv)(k) + \beta a.\end{aligned}$$

□

As such, we know that T has a fixed point. In order to get an idea behind the solution, let first give some simulation. We set $H = 15$ and consider a grid of fifteen values of $k = 1, 2, \ldots, 15$ and we specify $h(k) = k + 0.25(H - k)$ so every period the growth of the tree equals one fourth of the distance between its height and the maximal height.[51]

[51] I don't know if this is realistic but it certainly leads to a concave growth path.

```
%program for optimal tree cutting problem

%parameter values
par.beta    = 0.9;
par.eps     = 1e-6;
par.ngrids  = 15;

%state space
K = 1:ngrids;
%maximal height
H = K(ngrids);
%value functions
v    = zeros(1,ngrids);
Tv   = zeros(1,ngrids);
crit = 1;

while crit >= eps
    %next period height
    h_K = K + 0.25*(H - K);
    %updated value
    Tv = max(K, par.beta*interp1(K,v,h_K,'spline'));

    crit = max(abs(v - Tv));
    v = Tv;

end
```

The fixed point is given in Figure 4. We see that for low values of k, the fixed point $v(k)$ is above the diagonal, which means that the tree will not be cut. For high values of k, we have that $v(k) = k$, which means that the tree will be cut. As such, there seems to be a unique cutoff height k^* that determines the minimal height for the tree to be cut.

The cutoff point k^* should satisfy the indifference condition,

$$k^* = \beta v(h(k^*)).$$

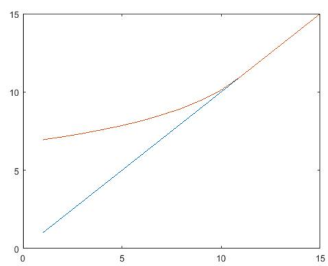

Figure 4: Value function and main diagonal.

In addition $h(k^*) \geq k^*$ so we know that (if we adhere to the conjecture), for height $k' = h(k^*)$, the tree will also be cut. This means that $k^* = \beta v(k') = \beta k' = \beta h(k^*)$. As such,

$$k^* = \beta h(k^*),$$
$$\rightarrow \frac{h(k^*)}{k^*} = 1/\beta.$$

The left hand side gives the proportional growth of a tree of height k^*. The right hand side gives the interest rate $(1+r) = 1/\beta$. If the left hand side is greater than the right hand side, it will be optimal not to cut the tree. Otherwise, cutting is optimal.

The following puts an assumption on the function $h(k)$, requiring that it cuts the line k/β at a single point k^* from above.

Assumption 1. *Assume that there is a $k^* \in [0, H]$ such that,*

- *if $k > k^*$ then $h(k) < \frac{1}{\beta}k$*
- *if $k < k^*$ then $h(k) > \frac{1}{\beta}k$.*

Lemma 9. *If assumption 1 is satisfied, then $v(k) = k$ for all $k \geq k^*$.*

Proof. The operator T with $(Tv) = \max\{k, \beta v(h(k))\}$ is a contraction mapping. Let

$$C = \{v \in B([0, H]) : \forall k \geq k^* : v(k) = k\}.$$

If we can show that $T(C) \subseteq C$, we know that the fixed point of T should be in the set C. So let $v \in C$ then we need to show that for all $k \geq k^*$, $(Tv)(k) = k$. Assume that $k \geq k^*$. Then,

$$(Tv)(k) = \max\{k, \beta v(h(k))\}.$$

we know that $h(k) \geq k \geq k^*$, so $v(h(k)) = h(k)$. This gives,

$$(Tv)(k) = \max\{k, \beta h(k)\}.$$

As $k \geq k^*$ we also know that, by assumption, $h(k) < k/\beta$, so,

$$(Tv)(k) = \max\{k, \beta h(k)\} \leq \max\{k, k\} = k.$$

From this it follows that $k \leq \max\{k, \beta v(h(k)\} = (Tv)(k) \leq k$ which means that $(Tv)(k) = k$ as was to be shown. □

Lemma 10. *If assumption 1 is satisfied, then for all $k \leq k^* : \beta v(h(k)) \geq k$.*

Proof. Let us first show that the fixed point v is a non-decreasing function, if $k \geq k'$ then $v(k) \geq v(k')$. Let $C = \{v \in B([0, H]) :$

v is increasing$\}$. In order to show that v is non-decreasing, it suffices to show that $T(C) \subseteq C$. As such, let $v \in C$. Then, if $k \geq k'$,

$$(Tv)(k) = \max\{k, \beta v(h(k))\} \geq \max\{k', \beta v(h(k'))\} = (Tv)(k').$$

which establishes the proof, $(Tv) \in C$.

Let

$$D = \{v \in B([0, H]) : v \text{ is nondecreasing and } \forall k \leq k^*, \beta v(h(k)) \geq k\}.$$

and let

$$D' = \{v \in B([0, H]) : f \text{ is nondecreasing and } \forall k < k^* : \beta v(h(k)) > k\}.$$

If we can show that $T(D) \subseteq D'$, we know that the fixed point of T should be in the set D'.

So let $f \in D$ (i.e. f is non-decreasing and for all $k \leq k^*$, $\beta v(h(k)) \geq k$) then we need to show that (Tv) is non-decreasing and $k < k^*$, $\beta(Tv)((h(k)) > k$. Above, we already showed that T maps non-decreasing functions to non-decrasing functions. For the second part, let $k < k^*$. We need to show that $\beta(Tv)(h(k)) > k$. Now,

$$(Tv)(h(k)) = \max\{h(k), \beta v(h(h(k)))\} \geq \max\{h(k), \beta v(h(k))\}.$$

The inequality follows from the fact that v is a non-decreasing function, so $h(h(k)) \geq h(k)$ implies $v(h(h(k))) \geq v(h(k))$. As $f \in D$, we also know that $\beta v(h(k)) \geq k$, so

$$(Tv)(h(k)) \geq \max\{h(k), \beta v(h(k))\} \geq \max\{h(k), k\} = h(k).$$

Finally given that $k < k^*$, by assumption 1 we know that $h(k) > k/\beta$, so

$$(Tv)(h(k)) \geq h(k) > k/\beta.$$

which means that $\beta(Tv)(h(k)) > k$, so $(Tv) \in D'$. □

Above two lemmata show that if assumption 1 is satisfied. Then there is a unique $k^*(= h(k^*)/\beta)$ such that for all $k < k^*$ the tree is not cut and for all $k \geq k^*$ the tree will be cut.

Optimal policy business cycles

The effectiveness of monetary economic policy depends on the expectations of the agents in the economy. Assume that the deviation of y_t which is the log of output from its natural level y^* is given by,

This model is borrowed from Ginsburgh and Michel, 1998, Optimal policy business cycles, Journal of Economic Dynamics and Control.

$$(y_t - y^*) = \gamma(\pi_t - \pi_t^e),$$

where π_t and π_t^e are the actual and expected rate on inflation in period t. Here $\gamma > 0$. This claims that only unexpected inflation can push output above its natural level. The policy maker's objective in each period is given by,

$$g(y_t, \pi_t) = \alpha(y_t - y^*) - \frac{\pi_t^2}{2} = \alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2}.$$

The government has a discount factor δ so the problem is to maximize

$$\sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2} \right).$$

If agents have rational expectations then $\pi_t^e = \pi_t$, so $y_t = y^*$ and the optimal policy is to set $\pi_t = 0$ at every point in time. On the other hand, if agents use a rule of thumb[52] we have,

$$\pi_{t+1}^a = \lambda \pi_t + (1-\lambda)\pi_t^a,$$

where $\lambda \in (0, 1]$. There is on the other hand, an intermediate case where a proportion x_t of agents use rational expectations while a fraction $(1 - x_t)$ form adaptive expectations. We assume that the average expected rate of inflation is a weighted average of the rates expected by rational and adaptive agents:

$$\pi_t^e = x\pi_t + (1-x)\pi_t^a.$$

then,

$$\begin{aligned} &\sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma(\pi_t - \pi_t^e) - \frac{\pi_t^2}{2} \right), \\ &= \sum_{t=0}^{\infty} \delta^t \left(\alpha\gamma\left((1-x)(\pi_t - \pi_t^a)\right) - \delta^t \frac{\pi_t^2}{2} \right). \end{aligned}$$

The Bellman equation is,

$$v(\pi_t^a) = \max_{\pi} \left\{ \alpha\gamma((1-x)(\pi_t - \pi_t^a) - \frac{\pi_t^2}{2} + \delta v(\lambda\pi_t + (1-\lambda)\pi_t^a) \right\}.$$

The Euler conditions are,

$$\begin{aligned} \pi_t &= \beta(1-x) + \delta q_{t+1}\lambda, \\ q_t &= -\beta(1-x) + \delta(1-\lambda)q_{t+1}. \end{aligned}$$

where $\beta = \alpha\gamma$ and q_t is the derivative of $v(\pi_{t+1}^a)$. So,

$$\begin{aligned} &\frac{\pi_{t-1} - \beta(1-x)}{\delta\lambda} = -\beta(1-x) + \delta(1-\lambda)\frac{\pi_t - \beta(1-x)}{\delta\lambda}, \\ &\leftrightarrow \pi_{t-1} - \beta(1-x) = -\beta(1-x)\delta\lambda + \delta(1-\lambda)\pi_t - \beta(1-x)\delta(1-\lambda), \\ &\leftrightarrow \pi_{t-1} - \delta(1-\lambda)\pi_t = \beta(1-x)(1-\delta), \\ &\leftrightarrow \pi_t - \frac{\pi_{t-1}}{\delta(1-\lambda)} = -\frac{\beta(1-x)(1-\delta)}{\delta(1-\lambda)}. \end{aligned}$$

[52] For example, they form expectations that are adaptive.

This is an explosive difference equation, so the only solution is the one at the steady state, where

$$\pi^* = \frac{\beta(1-x)(1-\delta)}{1-\delta(1-\lambda)}.$$

Now let us endogeneize the share of rational agents x_t. Assume that at time t decisions are made at no cost on the basis of adaptive expectations π_t^a. An agent θ can modify this decision at a fixed cost c using the new information π_t. There is a continuum of agents $\theta \in [0,1]$. Agent θ loses $\theta(\pi_t^a - \pi_t)^2$ when he uses π_t^a instead of π_t. let $\underline{\theta}_t$ be defined by,

$$\underline{\theta}_t(\pi_t^a - \pi_t)^2 = c.$$

The loss of agent θ is larger than c if $\theta \geq \underline{\theta}_t$. and the proportion x_t of agents that decide to change their decision is,

$$x_t = x(\pi_t^a, \pi_t) = \max\{0, 1 - c(\pi_t^a - \pi_t)^{-2})\}.$$

The Bellman equation is now,

$$v(\pi_t^a) = \max_{\pi_t} \left\{ \beta(1-x_t)(\pi_t - \pi_t^a) - \frac{\pi_t^2}{2} + \delta v(\lambda\pi_t + (1-\lambda)\pi_t^a) \right\},$$
$$\text{s.t. } x_t = \max\{0, 1 - c(\pi^a - \pi_t)^{-2}\}.$$

The model is a bit daunting to analyze analytically, so we will resort to a simulation exercise. The Matlab code for the Howard improvement algorithm is given below.

```

%parameters

beta     = 0.1;
c        = 0.0001;
pimax    = 0.05; %maximal value of state
pimin    = -0.1; %minimal value for state
ngrid    = 1000; %number of grid points
lambda   = 0.75;
delta    = 0.95;

Pia      = linspace(pimin,pimax,ngrid)'; %state space
Pi       = linspace(pimin,pimax,ngrid)';  %grid for optimal action
eps      = 1e-6;

%initializing
crit     = 1;
v        = zeros(ngrid,1); %value function
g        = zeros(ngrid,1); %best response function
Tv       = zeros(ngrid,1); %Bellman operator

while crit > eps
    for i = 1:ngrid
```

```
        %value of state var
        pi_a = Pia(i);
        %proportion of adaptive agents
        x = max(0,1-c*(pi_a - Pi).^(-2));

        %payoff
        psi = beta*(1-x).*(Pi-pi_a)-(Pi.^2)*(1/2);

        %v(lambda pi + (1-lambda) pi_a);
        Pinext = lambda*Pi+(1-lambda)*pi_a;
        vnext = interp1(Pia,v,Pinext,'spline');

        %Bellman iterator
        [Tv(i),g(i)] = max(psi+delta*vnext);
        end

    %best response
    Pi_opt = Pi(g);

    %initializing for the Howard iteration
    crit_2  = 1;
    Th      = zeros(ngrid,1);
    Th_0    = Tv; %initializing for Howard improvement
    while crit_2 > eps
        %proportion of adaptive agents
        X = max(0,1-c*(Pia-Pi_opt).^(-2));
        %value function
        Psi = beta*(1-X).*(Pi_opt-Pia)-(Pi_opt.^2)*(1/2);
        Vnext = interp1(Pia,Th_0, lambda*Pi_opt+(1-lambda)*Pia, 'spline');

        %policy function iteration
        Th = Psi + delta*Vnext;
        crit_2 = max(abs(Th - Th_0));
        Th_0 = Th;
    end

    crit    = max(abs(Tv-v));
    v       = Th_0;
    disp(crit);
    end

plot([pimin,pimax],[pimin,pimax], Pia, lambda*Pi_opt+(1-lambda)*Pia)
```

The algorithm finishes in 12 iterations in 5.85 seconds. Figure 5 plots π^a_{t+1} against the value of π^a_t. The stable state is situated at the point where the curve intersects with the diagonal. One sees that below the steady state the best response is above the diagonal. So, π^a increases over time. Suddenly the best response drops to below the diagonal. This shows that π^a will show cyclical behaviour. It will gradually increase and then suddenly drop to a lower value after which it will start increasing again.

Figure 6 shows a the evolution of inflation over time. Here the cyclical behaviour is clearly visible. Here, we have cycles of length 6. In 5 periods, inflation increases stepwise. In the sixth period inflation drops again to its starting value.

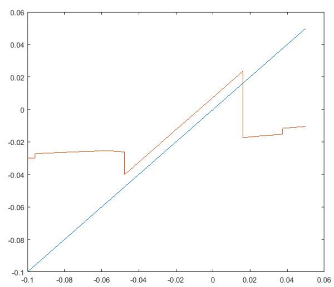

Figure 5: Value of π^a_{t+1} against the value of π^a_t.

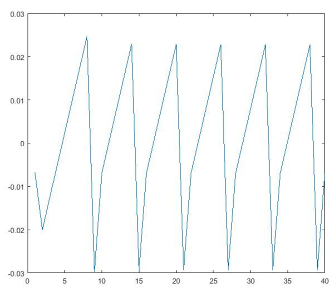

Figure 6: Evolution of π over time.

Stochastic dynamic programming

Intuitively, a stochastic dynamic program has the same components as a deterministic one. The only change is that the transition of one state to another is no longer certain. When transitions occur with a certain probability, states and decisions today lead to a distribution over possible states in the future.

Let X be the set of endogenous (chosen) states. These states are under the control of the decision maker. Now, we complement this with a set of exogenous states Z. Let $\Omega = X \times Z$ the set of states. A state $\omega = (x, z)$ has an endogeneous component, x and an exogenous component z. We define a correspondence $\Gamma : \Omega \twoheadrightarrow X$ such that $\Gamma(x, z)$ are the next period's endogenous states that the decision maker can choose if the current state is $\omega = (x, z)$. Let $A = \{(x, z, a) \in \Omega \times X : a \in \Gamma(x, z)\}$ be the graph of Γ and let $F : A \to \mathbb{R}$ be the instantaneous payoff function.[53] As usual, there is also a discount rate $\beta \in (0, 1)$. So far, this has been a simple extension of the usual framework. Now we add an additional element, namely that the state transition process is random. We will model this using a Markov-process. A Markov process is is determined by transition probabilities between periods. In essence it describes the probability of next period's state conditional on the state today. In particular, we consider a real valued function,

[53] Observe that now the instantaneous payoff function may depend on the current exogenous state, but not on the next period's exogenous state.

$$Q(A; x, z, a),$$

That gives the probability that the next periods state is in the set A if the current state is (x, z) and the choice for the next endogenous state is $a \in \Gamma(x, z)$.[54] If $f(\omega) : \Omega \to \mathbb{R}$ is a payoff function then the expected value of f tomorrow if (x, z) is the current state and $a \in \Gamma(x, z)$ is chosen is given by,

[54] Formally, $Q(A, ; x, z, a)$ is defined on a set $\mathcal{F} \times \Omega \times X$ where $\mathcal{F}$ is a sigma-algebra on Ω. The function $Q(., x, z, a)$ is a probability measure on $(\Omega, \mathcal{F})$ while $Q(A, ., ., a)$ should be a measurable function on $(\Omega, \mathcal{F})$.

$$\int f(\omega')Q(d\omega', x, z, a).$$

A policy function $h : \Omega \to X$ determines for each state $\omega = (x, z) \in \Omega$ an endogenous variable $h(x, z) \in \Gamma(x, z)$. Assume that the current state is $\omega_0 = (x_0, z_0)$. Then if the decision maker follows the policy rule h, the next period's expected payoff is given by,

$$E_0[\beta F(\omega_1, h(\omega_1))|\omega_0] = \int \beta F(\omega_1, h(\omega_1))Q(d\omega_1; \omega_0, h(\omega_0)).$$

The expected payoff within two periods F two periods from now is given by,

$$\begin{aligned} &E_0[\beta^2 F(\omega_2, h(\omega_2))|\omega_0], \\ &= \int \left[\int \beta^2 F(\omega_2, h(\omega_2))Q(d\omega_2; \omega_1, h(\omega_1)) \right] Q(d\omega_1; \omega_0, h(\omega_0)), \end{aligned}$$

The expected payoff for the first n periods is then determined by,

$$\begin{aligned} u_n(h) &= E_0 \left[\sum_{t=1}^{n} \beta^t F(\omega_t, h(\omega_t)) \middle| \omega_0 \right], \\ &= F(\omega_0, h(\omega_0)) + \int \beta F(\omega_1, h(\omega_1))Q(d\omega_1, \omega_0, h(\omega(0))), \\ &+ \int \left[\int \beta^2 F(\omega_2, h(\omega_2))Q(d\omega_2; \omega_1, h(\omega_1)) \right] Q(d\omega_1; \omega_0, h(\omega_0)), \\ &+ \ldots, \\ &+ \int \ldots \int \beta^n F(\omega_n, h(\omega_n))Q(d\omega_n; \omega_{n-1}, h(\omega_{n-1})) \ldots Q(d\omega_1; \omega_0, h(\omega_0)). \end{aligned}$$

Let $u_\infty(h) = \lim_n u_n(h)$. The aim of the decision maker is to find a policy function h to maximize $u_\infty(h)$,[55]

$$\sup_h u_\infty(h).$$

[55] Observe that we have not showed yet that this maximization problem is well defined.

The aim of this chapter is to relate the solution of this problem (if it exists) to the solution of the following functional equation,

$$\begin{aligned} v(\omega) &= \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + \int \beta v(\omega')Q(d\omega'; \omega, a) \right\}, \\ &= \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + E\left[v(\omega')|\omega, a \right] \right\} \end{aligned}$$

This is the Bellman equation for the stochastic problem. It is related to the following Bellman operator T,

$$(Tv)(\omega) = \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + \int \beta v(\omega')Q(d\omega'; \omega, a) \right\},$$

Definition 14 (regularity). *The stochastic dynamic programming problem $(\Omega, \Gamma, F, \beta, Q)$ is regular if the state function $F : A \to \mathbb{R}$ is continuous, where $A = \{(\omega, a) : a \in \Gamma(\omega)\}$, the transition function $\Gamma : \Omega \to X$ is non-empty continuous and compact valued and there exists a continuous function $\phi : \Omega \to \mathbb{R}_{++}$ such that,*

1. *There exists an* $M \geq 0$ *such that for all* $\omega \in \Omega$,

$$\max_{a \in \Gamma(\omega)} |F(\omega, a)| \leq M\phi(\omega).$$

2. *There exists a* $\theta \in (0, 1)$ *such that for all* $\omega \in \Omega$,

$$\beta \max_{a \in \Gamma(\omega)} \int \phi(\omega')Q(d\omega'; \omega, a) \leq \theta\phi(\omega).$$

3. *for any* $(\omega, a) \in A$,

$$\int \phi(\omega')Q(d\omega'; \omega, a) < \infty.$$

4. *If* $f : \Omega \to \mathbb{R}$ *is continuous and* $f \in B_\phi(\Omega)$*, then*

$$R(\omega, a) = \int f(\omega')Q(d\omega', \omega, a),$$

is a bounded continuous function on A.

Condition 4 is called the Feller condition.

Theorem 13. *If the problem* $(\Omega, \Gamma, F, \beta, Q)$ *is regular then the Bellman operator maps* $B_\phi(\Omega)$ *into* $B_\phi(\Omega)$ *and is a contraction mapping.*

Proof. Let $v \in B_\phi(\Omega)$ then, there is an $M > 0$ such that,

$$\begin{aligned} \int |v(\omega')|Q(d\omega; \omega, a) &\leq \int M\phi(\omega')Q(d\omega'; \omega, a), \\ &\leq \infty \end{aligned}$$

This shows that v is integrable. Let us show that T maps $B_\phi(\Omega)$ into $B_\phi(\Omega)$. If $v \in B_\phi(\Omega)$ then Tv is continuous by the theorem of the maximum. To see that (Tv) is bounded in the $\|.\|_\phi$ norm, observe that,

$$\begin{aligned} |(Tv)(\omega)| &= \left| \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + \beta \int v(\omega')Q(d\omega'; \omega, a) \right\} \right|, \\ &\leq \max_{a \in \Gamma(\omega)} |F(\omega, a)| + \beta \max_{a \in \Gamma(\omega)} \left\{ \int |v(\omega')|Q(d\omega'; \omega, a) \right\}, \\ &\leq \max_{a \in \Gamma(\omega)} |F(\omega, a)| + \beta \max_{a \in \Gamma(\omega)} \left\{ \int \|v\|_\phi \phi(\omega')Q(d\omega'; \omega, a) \right\}, \\ &\leq M\phi(\omega) + \beta\|v\|_\phi \theta\phi(\omega), \\ &= (M + \|v\|_\phi \theta)\phi(\omega). \end{aligned}$$

so $\|(Tv)\|_\phi$ is bounded by $M + \|v\|_\phi \theta$ which is finite.

For a contraction mapping, we verify Blackwell's conditions. For monotonicity, let $v \geq w$ then

$$\begin{aligned} (Tv)(\omega) &= \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + \beta \int v(\omega')Q(d\omega'; \omega, a) \right\}, \\ &\geq \max_{a \in \Gamma(\omega)} \left\{ F(\omega, a) + \beta \int w(\omega')Q(d\omega'; \omega, a) \right\}, \\ &= (Tw)(\omega). \end{aligned}$$

For additivity,

$$
\begin{aligned}
(T(v+\alpha\phi))(\omega) &= \max_{a\in\Gamma(\omega)} \left\{ F(\omega,y) + \beta \int (v+\alpha\phi)(\omega')Q(d\omega';\omega,a) \right\}, \\
&\leq (Tv)(\omega) + \beta\alpha \max_{a\in\Gamma(\omega)} \int \phi(\omega')Q(d\omega;\omega,a), \\
&\leq (Tv)(\omega) + \theta\alpha\phi(\omega),
\end{aligned}
$$

as was to be shown. □

Now, let's go back to our original problem,

$$\sup_h u_\infty(h).$$

We will relate the solution to this problem with the fixed point of the Bellman operator.

Lemma 11. *Let $(\Omega, F, \Gamma, \beta, Q)$ be a regular problem. Let h be a policy function. Then $u_\infty(h)$ exists and the set $\{u_\infty(h)\}$ is bounded from above, so the* sup *problem is well defined.*

Proof. For a given h, we have that,

$$
\begin{aligned}
u_n(h) = {} & F(\omega_0, h(\omega_0)) +, \\
& + \beta \int F(\omega_1, h(\omega_1))Q(d\omega_1;\omega_0,h(\omega_0)), \\
& + \beta \int \left[\int F(\omega_2, h(\omega_2)Q(d\omega_2;\omega_1,h(\omega_1)) \right] Q(d\omega_1;\omega_0,h(\omega_0)), \\
& + \ldots, \\
& + \beta^n \int \ldots \int F(\omega_n, h(\omega_n))Q(d\omega_n;\omega_{n-1},h(\omega_{n-1}))\ldots Q(d\omega_1;\omega_0,h(\omega_0)).
\end{aligned}
$$

Take any term in this summation and take the innermost integral,

$$
\begin{aligned}
\int \beta^t F(\omega_t, h(\omega_t))Q(d\omega_t;\omega_{t-1},h(\omega_{t-1})) &\leq \beta^t \int |F(\omega_t,h(\omega_t))| Q(d\omega_t;\omega_{t-1},h(\omega_{t-1})), \\
&\leq \beta^t \int M\phi(\omega_t)Q(d\omega_t;\omega_{t-1},h(\omega_{t-1})), \\
&\leq \beta^t M \frac{\theta}{\beta}\phi(\omega_{t-1})
\end{aligned}
$$

Then taking the two inner integrals,

$$
\begin{aligned}
& \int \left[\int \beta^t F(\omega_t, h(\omega_t))Q(d\omega_t;\omega_{t-1},h(\omega_{t-1})) \right] Q(d\omega_{t-1};\omega_{t-2},h(\omega_{t-2})), \\
& \leq \beta^t \int M \frac{\theta}{\beta}\phi(\omega_{t-1})d(\omega_{t-1};\omega_{t-2},h(\omega_{t-2})), \\
& \leq M\beta^t \frac{\theta^2}{\beta^2}\phi(\omega_{t-2}).
\end{aligned}
$$

Iterating further gives that the $t-$th term is bounded from above by $M\theta^t\phi(\omega_0)$. Doing this for every term gives,

$$\begin{aligned} u_n(h) &\leq M\phi(\omega_0)(1+\theta+\theta^2+\ldots,+\theta^n), \\ &= M\phi(\omega_0)\frac{1-\theta^{n+1}}{1-\theta}. \end{aligned}$$

Letting $n \to \infty$, we see that $u_n(h) \leq M\phi(\omega_0)/(1-\theta)$. □

Next, let us show that the fixed point of the Bellman operator is greater than any $u_\infty(h)$.

Lemma 12. *Let $(\Omega, F, \Gamma, \beta, Q)$ be a regular problem. Let h be a policy function and let v be the fixed point of the Bellman operator, then $v(\omega_0) \geq u_\infty(h)$.*

Proof. We have that,

$$\begin{aligned} v(\omega_0) &\geq F(\omega_0, h(\omega_0)) + \beta\int v(\omega_1)Q(d\omega_1;\omega_0,h(\omega_0)), \\ &\geq F(\omega_0, h(\omega_0)) + \beta\int F(\omega_1,h(\omega_1))Q(d\omega;\omega_0,h(\omega_0)), \\ &+\beta^2\int\int v(\omega_2)Q(d\omega_2;\omega_1,h(\omega_1))Q(d\omega_1;\omega_0,h(\omega_0)). \qquad = \ldots, \\ &= u_n(h)+\beta^{n+1}\int \ldots \int v(\omega_{n+1})Q(d\omega_{n+1};\omega_n,h(\omega_n))\ldots,Q(d\omega_1;\omega_0,h(\omega_0)). \end{aligned}$$

Taking the limit to infinity, the first term goes to $u_\infty(h)$. So we only need to show that the second term goes to zero. However, the inner integral is bounded by,

$$\begin{aligned} &\beta^{n+1}\int v(\omega_{n+1})Q(d\omega_{n+1};\omega_n,h(\omega_n)), \\ &\leq \beta^{n+1}\|v\|_\phi\int \phi(\omega_{n+1})Q(d\omega_{n+1};\omega_n,h(\omega_n)), \\ &\leq \beta^{n+1}\|v\|_\phi\frac{\theta}{\beta}\phi(\omega_n) \end{aligned}$$

Iterating further over all other integrations gives finally, that the term is bounded from above by,

$$\|v\|_\phi\theta^{n+1}\phi(\omega_0)$$

This goes to zero as $n \to \infty$. □

For $\omega \in \Omega$ define the value $g(\omega)$ as,

$$g(\omega) \in \arg\max_{a\in\Gamma(\omega)}\left\{F(\omega,a)+\beta\int v(\omega')Q(d\omega';\omega;a)\right\}.$$

Lemma 13. *Let $(\Omega, F, \Gamma, \beta, Q)$ be a regular problem, then $v(\omega_0) = u_\infty(g)$.*

Proof. We have that,

$$
\begin{aligned}
v(\omega_0) &= F(\omega_0, g(\omega_0)) + \beta \int v(\omega_1) Q(d\omega_1; \omega_0, g(\omega_0)),\\
&= F(\omega_0, g(\omega_0)) + \beta \int F(\omega_1, g(\omega_1)) Q(d\omega; \omega_0, g(\omega_0)),\\
&+ \beta^2 \int \int v(\omega_2) Q(d\omega_2; \omega_1, g(\omega_1)) Q(d\omega_1; \omega_0, g(\omega_0)). \qquad = \dots,\\
&= u_n(g) + \beta^{n+1} \int ... \int v(\omega_{n+1}) Q(d\omega_{n+1}; \omega_n, g(\omega_n)) \dots, Q(d\omega_1; \omega_0, g(\omega_0)).
\end{aligned}
$$

Taking the limit to infinity, the first term goes to $u_\infty(g)$. So we only need to show that the second term goes to zero. However, the inner integral is bounded by,

$$
\begin{aligned}
&\beta^{n+1} \int v(\omega_{n+1}) Q(d\omega_{n+1}; \omega_n, g(\omega_n)),\\
&\leq \beta^{n+1} \|v\|_\phi \int \phi(\omega_{n+1}) Q(d\omega_{n+1}; \omega_n, g(\omega_n)),\\
&\leq \beta^{n+1} \|v\|_\phi \frac{\theta}{\beta} \phi(\omega_n)
\end{aligned}
$$

Iterating further over all other integrations gives finally, that the term is bounded from above by,

$$
\|v\|_\phi \theta^{n+1} \phi(\omega_0)
$$

This goes to zero as $n \to \infty$. □

Simulations for models of uncertainty

LET US FIRST look at a very simple model of optimal growth with stochastic shocks. As usual, we take the utility of the consumer to be $u(c) = \ln(c)$. Output is produced using outputs in the previous period net of consumption. In particular, the output in period $t+1$ is given by,

$$y_{t+1} = \eta_{t+1}(y_t - c_t)^\alpha,$$

where η_{t+1} is a stochastic shock with distribution function F, realized in period $t+1$. We assume that η_t is i.i.d.

$$\begin{aligned}
&\sum_{t=0}^{\infty} E_0 \left(\beta^t \max_{c_t} \ln(c_t) \right), \\
\text{s.t.} & c_t \leq y_t, \\
& y_{t+1} = \eta_{t+1}(y_t - c_t)^\alpha.
\end{aligned}$$

This gives the Bellman equation,

$$v(y_t) = \max_{c_t \leq y_t} \left(\ln(c_t) + \beta \int v(\eta(y_t - c_t)^\alpha P(d\eta_{t+1}) \right).$$

Simulation of this model is analogues as for the case under certainty. The only difference is here to estimate the integral. This can be done using Monte-Carlo simulation.

- Draw a large number of random variables $\eta_1, \ldots, \eta_N$ according to the distribution P.
- Compute the mean,

$$\frac{1}{N} \sum_{n=1}^{N} v(\eta_n(y_t - c_t)^\alpha).$$

For the algorithm, it is important to draw the values of η_n before entering the loop on the function value iteration. This guarantees the convergence of the algorithm. If you draw for each loop new random values, convergence is not guaranteed.

The following gives the code. We assume that η has value $e^{\mu+s\varepsilon}$ where ε has a standard normal distribution.

```
%parameter values
par.alpha = 0.4;
par.beta = 0.96;
par.mu = 0;
par.s = 0.1;

gridsize = 100; %for the output values
shocksize = 1000; %for the Monte Carlo integration

crit = 1;
eps = 10e-6;

v = ones(gridsize,1);
Y = linspace(0.1,7,gridsize)';
g = ones(gridsize,1);
Tv = zeros(gridsize,1);
HV = zeros(gridsize,1);

shocks = exp(par.mu + par.s*randn(1,shocksize));%draw the shocks

while crit>=eps
    %value function iteration
    for i = 1:gridsize
        y = Y(i);
        C = Y(1:i); %feasible consumption values
        Ynext = y - C;
        temp1 = (Ynext.^par.alpha)*shocks; %get eta (y - c)^alpha
        %use linear interpolation
        temp2 = interp1(Y,v,temp1,'linear');
        temp3 = mean(temp2,2); %Monte carlo integration

        [Tv(i), g(i)] = max(log(C) + par.beta*temp3);%find optimal c
    end

    %policy function interation
    crit2 = 1;
    while crit2>eps
        Ynext = Y - Y(g);
        temp1 = (Ynext.^par.alpha)*shocks;
        temp2 = interp1(Y,Tv,temp1,'linear');
        temp3 = mean(temp2,2);
        Hv = log(Y(g)) + par.beta*temp3;


        crit2 = max(abs(Hv-Tv)); %distance
        Tv = Hv; %update
    end

    crit = max(abs(Tv-v)); %compute distance
    disp(crit);
    v = Tv;%update
end
```

The previous example was rather easy in the sense that the value function (and policy function) where independent of the stochastic component. In other examples, however, this is no longer the case. Let's consider a growth model where $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ and,

$$k_{t+1} = e^a k_t^\alpha - c_t + (1-\delta)k_t.$$

Here a is a stochastic variable that takes on two possible values a_1 and a_2. The transition probability between the two states is given by a Markov transition matrix,

$$\Pi = \begin{bmatrix} \pi_1 & 1-\pi_1 \\ 1-\pi_2 & \pi_2 \end{bmatrix}.$$

Here π_i is the probability of being in a_i next period, given that a_i is the current state. The optimization problem is then,

$$\begin{aligned} &\max_{t=1}^{\infty} E_0 \left(\beta^t \frac{c_t^{1-\sigma}-1}{1-\sigma} \right), \\ \text{s.t. } & k_{t+1} = e^a k_t^\alpha - c_t + (1-\delta)k_t, \\ & \Pr(a_i|a_i) = \pi_i. \end{aligned}$$

In terms of the Bellman equation, we have,

$$v(k,a) = \max_{c_t \le e^a k_t^\alpha + (1-\delta)k_t} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \beta \sum_{i=1}^{2} \Pr(a_i|a) v(e^{a_i}k^\alpha - c + (1-\delta)k, a_i) \right\}.$$

Actually, we have two policy functions $v(k,a_1)$ and $v(k,a_2)$ which may differ as the state might be different.

The next program simulates the optimal value functions using policy function iteration.

```
%parameter values
par.sigma = 1.5;
par.delta = 0.10;
par.beta = 0.95;
par.alpha = 0.30;


gridsize = 1000;
crit=1;
eps = 1e-6;

%Markov chain
p = 0.9;
PI = [p 1-p; 1-p p];
A = [0.8 1.2];

kmin = 0.2;
kmax = 6;
K = linspace(kmin,kmax,gridsize);
c = zeros(gridsize,2);
v = zeros(gridsize,2);
g = zeros(gridsize,2);
```

```
Tv = zeros(gridsize,2);
Hv = zeros(gridsize,2);

while crit >eps
    for i = 1:gridsize
        %iterate over the states
        for state = 1:2
            k = K(i);
            c = A(state)*k^par.alpha + (1-par.delta)*k - K;
            %feasible values
            B = c>0;
            util(B==1,state) = (c(B).^(1-par.sigma)-1)/(1-par.sigma);
            util(B==0,state) = -inf;
        end

        %optimization
        [Tv(i,:), g(i,:)] = max(util+ par.beta*(v*PI));
    end

    %policy function iteration
    crit2 = 1;
    while crit2 > eps
        %optimal consumption bundles
        Cp = zeros(gridsize,2);
        %next period capital stock
        Knext = zeros(gridsize,2);
        Tvnext = zeros(gridsize,2);
        for state = 1:2
            Cp(:,state) = A(state)*K.^par.alpha + (1-par.delta)*K-K(g(:,state));
            util(:,state) = (Cp(:,state).^(1-par.sigma)-1)/(1-par.sigma);
            %next period v function if state is 1
            vnext(:,1) = Tv(g(:,state),1);
            %next period v function if state is 0
            vnext(:,2) = Tv(g(:,state),2);
            %expected next period v function
            Tvnext(:,state) = vnext(:,:)*PI(state,:)';
        end
        Hv = util + par.beta*Tvnext;
        crit2 = max(max(abs(Hv-Tv)));
        Tv = Hv;
    end

    crit = max(max(abs(Tv-v)));
    v = Tv;
    disp(crit);
    end
```

The program computes the optimal response for every level of capital and every state of the world. We can also simulate trajectories of capital. The following program provides the code to do this.

```
%Initialization
G_0 = 10;
state_0 = 1;

%number of periods
nperiods = 200;


```

```
Kap = zeros(nperiods,1);
S = zeros(nperiods,1);
G = zeros(nperiods,1);
C = zeros(nperiods,1);

%initial values
G(1) = G_0;
Kap(1) = K(G_0);
Kap(2) = K(G_0);
S(1) = state_0;
C(1) = 0;

for t = 2:nperiods-1

    r = rand();
    %determine state of the world
    if r <= PI(S(t-1),1);
        S(t) = 1;
    else
        S(t) = 2;
    end

    %policy function in period t
    G(t) = g(G(t-1),S(t));
    %capital stock in period t+1
    Kap(t+1,1) = K(G(t));
    %consumption in period t
    C(t) = A(S(t))*Kap(t)^par.alpha + (1-par.delta)*Kap(t) - Kap(t+1);
end
```

Applications

Consider the problem of a cake of size x that has to be eaten in its entirety in one single period. There is a taste shock z that takes on two possible values $0 < z_\ell < z_h$. Let $p_{\ell h}$ be the probability of tastes switching from z_ℓ to z_h and let $p_{h\ell}$ be the probability of switching from z_h to z_ℓ. Eating the cake gives a value of $zu(x)$. The value function can be written as follows,

$$v(z_\ell) = \max\{z_\ell u(x), \beta[p_{\ell h} v(z_h) + (1 - p_{\ell h}) v(z_\ell)]\}.$$
$$v(z_h) = \max\{z_h u(x), \beta[p_{h\ell} v(z_\ell) + (1 - p_{h\ell}) v(z_h)]\}.$$

where $\beta < 1$ is a discount rate. The first term between braces is the value of immediate consumption. The second term is the payoff of waiting for one period.

Consider the strategy that says to always wait. the optimal value function for this strategy gives a payoff of 0. As such, if $z_\ell u(x), z_h u(x) \leq 0$, then it is an optimal strategy to always wait, and the optimal value function $v(z_\ell) = v(z_h) = 0$. However, this is not a very nice solution. As such, we assume from now on that $z_h u(x) > 0$.

We characterize the solution to this problem in a number of steps,

- Step 1: There is at least one state $z = z_\ell$ or $z = z_h$ where the cake is eaten.
  If, towards a contradiction, the cake is never eaten, then

  $$v(z_\ell) = \beta[p_{\ell h} v(z_h) + (1 - p_{\ell h}) v(z_\ell)],$$
  $$v(z_h) = \beta[p_{h\ell} v(z_\ell) + (1 - p_{h\ell}) v(z_h)]$$

  If $v(z_\ell) = v(z_h) = v$ we get $v = \beta v$ which can only be if $v = 0$. However, $v(z_h) \geq z_h u(x) > 0$ which gives the desired contradiction.

  If $v(z_\ell) < v(z_h)$ then $v(z_h)$ is less than a weighted average of $v(z_\ell)$ and $v(z_h)$ again a contradiction which can only be if $v(z_h) = 0$, a contradiction. If $v(z_h) < v(z_\ell)$ then $v(z_\ell)$ is lower than a weighted average of both values which can only be if $v(z_\ell) = 0$. But then $v(z_h) = \beta(1 - p_{h\ell})v(z_h)$ which shows that $v(z_h) = 0$, again a contradiction.

- Step 2: If $z = z_h$ then the cake will be eaten, i.e. $v(z_h) = z_h u(x)$. Assume not, then at z_h the cake is not eaten. This means that $v(z_h) = \beta[p_{h\ell}v(z_\ell) + (1 - p_{h\ell})v(z_h)]$. By the first point above, we have that the cake must be eaten when $z = z_\ell$. Then $v(z_\ell) = z_\ell u(x)$. So,

$$\begin{aligned} z_h u(x) \leq v(z_h) &= \frac{\beta p_{h\ell}}{1 - \beta(1 - p_{h\ell})} z_\ell u(x), \\ \rightarrow z_h(1 - \beta) + z_h \beta p_{h\ell} &\leq \beta z_\ell p_{h\ell}. \end{aligned}$$

  But this is impossible as $z_h > z_\ell$. This shows that $v(z_h) = z_h u(x)$.

- Step 3: The cake will not eaten in $z = z_\ell$ if and only if $\frac{\beta p_{\ell h}}{1-\beta} \geq \frac{z_\ell}{z_h - z_\ell}$. The first two steps show that the cake will be eaten whenever $z = z_h$. As such, $v(z_h) = z_h u(x)$. The problem is to determine whether the cake will be eaten in the state z_ℓ. We have that,

$$v(z_\ell) = \max\{z_\ell u(x), \beta[p_{\ell h} z_h u(x) + (1 - p_{\ell h})v(z_\ell)]\}.$$

  Assume that the cake will not be eaten, then

$$v(z_\ell) = \frac{\beta p_{\ell h} z_h u(x)}{1 - \beta(1 - p_{\ell h})}.$$

  we need that this is larger than $z_\ell u(x)$ or

$$\begin{aligned} &\frac{\beta p_{\ell h} z_h u(x)}{1 - \beta(1 - p_{\ell h})} \geq z_\ell u(x), \\ \rightarrow &\frac{\beta p_{\ell h} z_h}{1 - \beta(1 - p_{\ell h})} \geq z_\ell, \\ \rightarrow &\frac{\beta p_{\ell h}}{1 - \beta} \geq \frac{z_\ell}{z_h - z_\ell}. \end{aligned}$$

  For the reverse, if the cake is eaten in $z = z_\ell$ then $v(z_\ell) = z_\ell u(x)$. and,

$$\begin{aligned} &z_\ell u(x) \geq \beta[p_{\ell h} z_h u(x) + (1 - p_{\ell h}) z_\ell u(x)], \\ \rightarrow &z_\ell \geq \beta[p_{\ell h} z_h + (1 - p_{\ell h}) z_\ell], \\ \rightarrow &z_\ell \geq \beta p_{\ell h}(z_h - z_\ell) + \beta z_\ell], \\ \rightarrow &\frac{z_\ell}{z_h - z_\ell} \geq \frac{\beta p_{\ell h}}{1 - \beta}. \end{aligned}$$

  which is the reverse condition.

Let us add a new twist to the problem. Assume that if the cake is not eaten today, then a fraction $(1-\delta)$ of the cake is lost. In order to solve this, we have to include the size of the cake into the state, which now becomes (x, z). Then

$$\begin{aligned} v(x, z_\ell) &= \max\{z_\ell u(x); \beta[p_{\ell h} v(\delta x, z_h) + (1-p_{\ell h}) v(\delta x, z_\ell)]\}, \\ v(x, z_h) &= \max\{z_h u(x); \beta[p_{h\ell} v(\delta x, z_\ell) + (1-p_{h\ell}) v(\delta x, z_h)]\}, \end{aligned}$$

- Let $u(x)$ be non-decreasing in x, then the function $v(., z)$ should also be non-decreasing in it's first argument.
  The proof is based on Lemma 1. Consider the bellman operator T associated with the functional equation above. It is easy to see that this is a contraction mapping. As such, it suffices to show that it maps non-decreasing functions to non-decreasing functions. Indeed, if $v(., z)$ is nondecreasing, and $x \geq x'$ then

  $$\begin{aligned} (Tv)(x, z_\ell) &= \max\{z_\ell u(x); \beta[p_{\ell h} v(\delta x, z_h) + (1-p_{\ell h}) v(\delta x, z_\ell)]\}, \\ &\geq \max\{z_\ell u(x'); \beta[p_{\ell h} v(\delta x', z_h) + (1-p_{\ell h}) v(\delta x', z_\ell)]\}, \\ &= (Tv)(x', z_\ell). \end{aligned}$$

  A similar reasoning holds for $v(., z_h)$.
- There is at least one state where the cake is eaten.
  The argument is similar as before.
- The cake will be eaten when $z = z_h$.
  The argument is similar as before.

The only problem is now to decide when the cake will be eaten in the $z = z_\ell$ states. Using the fact that $v(x, z_h) = z_h u(x)$, this will be the case whenever,

$$z_\ell u(x) \geq \beta[p_{\ell h} z_h u(\delta x) + (1-p_{\ell h}) v(\delta x, z_\ell)],$$

We will try to look for a cut-off level x^* that separates the region where it is optimal to eat the cake and where it is optimal not to eat the cake. Before we do so, it will be convenient make the additional assumption that $u(x) = \ln(x)$. Consider the following operator,

$$(T^* v)(x) = \beta[p_{\ell h} z_h \ln(\delta x) + (1-p_{\ell h}) v(\delta x)].$$

It is readily verified that T^* is a contraction mapping. From $B(X)_\phi$ to $B(X)_\phi$ where, for example, $\phi(x) = |\ln(x) + 1|$. As such, it has a unique fixed point, say v^*. Observe that v^* gives the value of the policy function that states to always wait.

To find the fixed point, we will guess a functional form and then verify that it indeed gives a solution. Given the log function on the right hand side, let us take an educated guess that $v^*(x) = A + B\ln(x)$. Then, we need that,

$$\begin{aligned} A + B\ln(x) &= \beta p_{\ell h} z_h \ln(\delta x) + \beta(1-p_{\ell h})(A + B\ln(\delta x)), \\ &= (\beta p_{\ell h} z_h + \beta(1-p_{\ell h})B)\ln(x) + \beta[p_{\ell h} z_h \ln(\delta) + (1-p_{\ell h})(A + B\ln(\delta))]. \end{aligned}$$

Equating coefficients gives,

$$\begin{aligned} B &= \frac{\beta p_{\ell h} z_h}{1-\beta(1-p_{\ell h})}, \\ A &= \frac{\beta \ln(\delta)}{1-\beta(1-p_{\ell h})}(p_{\ell h} z_h + (1-p_{\ell h})B) \end{aligned}$$

Now, let us return to our functional equation.

$$v(x, z_\ell) = \max\{z_\ell \ln(x), \beta[p_{\ell h} z_h \ln(\delta x) + (1-p_{\ell h})v(\delta x, z_\ell)]\}.$$

Assume that there is a level x^* such that if $x < x^*$ then it is optimal to wait and if $x \geq x^*$ then it is optimal to eat the cake.

Then if $x \leq x^*$, we must have that,

$$v(x, z_\ell) = \beta p_{\ell h} z_h \ln(\delta x) + \beta(1-p_{\ell h})v(\delta x, z_\ell).$$

We already know the solution to this functional equation, namely,[56]

$$v(x, z_\ell) = A + B\ln(x).$$

[56] This uses the fact that $\delta x \leq x \leq x^*$ so at δx, the optimal choice is also to wait.

If $x \geq x^*$ it is assumed that it is optimal to eat the cake, so we must have that for $x \geq x^*$

$$v(x, z_\ell) = z_\ell \ln(x).$$

At x^* the decision maker is assumed to be indifferent between eating and not, so

$$\begin{aligned} z_\ell \ln(x^*) &= A + B\ln(x^*), \\ \to \ln(x^*) &= \frac{A}{z_\ell - B}. \end{aligned}$$

In order for this solution to be the right one, we need to show two things. First, we need to show that if $x \leq x^*$ then

$$v(x) = A + B\ln(x) \geq z_\ell \ln(x).$$

Next we need to show that if $x \geq x^*$ then,

$$v(x) = z_\ell \ln(x) \geq \beta[p_{\ell h} z_h \ln(x) + (1-p_{\ell h})v(\delta x)].$$

Let's see:

- For all $x \leq x^*$: $A + B\ln(x) = v(x) \geq z_\ell \ln(x)$.
  For this to be true, we need the condition the slope of the expression on the left hand side is smaller than the one on the right hand side. So we need to impose the condition that $B < z_\ell$.

- For all $x \geq x^*$: $z_\ell \ln(x) = v(x) \geq \beta p_{\ell h} z_h \ln(\delta x) + \beta(1 - p_{\ell h})v(\delta x)$.
  There are two cases to consider.

  1. $\delta x \geq x^*$. In this case at δx, it is optimal to eat the cake, so $v(\delta x) = z_\ell \ln(\delta x)$. We need to show that,

  $$z_\ell \ln(x) \geq \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h}) z_\ell \ln(\delta x)],$$
  $$\leftrightarrow z_\ell \ln(x) \geq \frac{\beta p_{\ell h} z_h}{1 - \beta(1 - p_{\ell h})} \ln(x) + \frac{\beta \ln(\delta)}{1 - \beta(1 - p_{\ell h})}[p_{\ell h} z_h + (1 - p_{\ell h}) z_\ell].$$

  Now, $x \geq x^*$, so given that $B < z_\ell$,

  $$\begin{aligned} z_\ell \ln(x) &\geq B\ln(x) + A, \\ &= \frac{\beta p_{\ell h} z_h}{1 - \beta(1 - p_{\ell h}) z_\ell} \ln(x) + \frac{\beta \ln(\delta)}{1 - \beta(1 - p_{\ell h})}(p_{\ell h} z_h + (1 - p_{\ell h})B). \end{aligned}$$

  The second term on the right hand side is negative, as $\ln(\delta) < 0$ also, $B \leq z_\ell$ by assumption. As such,

  $$z_\ell \ln(x) \geq \frac{\beta p_{\ell h} z_h}{1 - \beta(1 - p_{\ell h}) z_\ell} \ln(x) + \frac{\beta \ln(\delta)}{1 - \beta(1 - p_{\ell h})}(p_{\ell h} z_h + (1 - p_{\ell h}) z_\ell).$$

  as we needed to show.

  2. $\delta x < x^*$. In this case, it is optimal not the eat the cake at δx, so $v(\delta x) = A + B\ln(\delta x)$ and we need to show that,

  $$z_\ell \ln(x) \geq \beta[p_{h\ell} z_h \ln(\delta x) + (1 - p_{h\ell})(A + B\ln(\delta x))].$$

  Now, given that $x \geq x^*$,

  $$\begin{aligned} z_\ell \ln(x) &\geq A + B\ln(x), \\ &= \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h})(A + B\ln(\delta x))]. \end{aligned}$$

  The second line follows from the definition of A and B given that $A + B\ln(x)$ is the unique fixed point of the operator

  $$(T^* v)(x) = \beta[p_{h\ell} z_h \ln(\delta x) + (1 - p_{\ell h})v(\delta x)].$$

This shows the following lemma.

Lemma 14. *If $B < z_\ell$ then there is a unique point x^* where*

$$\ln(x^*) = \frac{A}{B - z_\ell},$$

When $z = z_\ell$ then it is optimal to wait when $x \leq x^*$ and optimal to eat the cake when $x \geq x^*$.

Now it is also possible that $B > z_\ell$. Then the optimal solution looks the reverse: there is some level $\tilde{x}$ such that if $x \geq \tilde{x}$ it is optimal to wait. If $x \leq \tilde{x}$ it is optimal to eat the cake. For this, we need to look at the condition,

$$\begin{aligned} z_\ell \ln(\tilde{x}) &= \beta[p_{\ell h} z_h \ln(\delta \tilde{x}) + (1 - p_{\ell h}) z_\ell \ln(\delta \tilde{x})], \\ \leftrightarrow z_\ell \ln(\tilde{x}) &= \tilde{A} + B \ln(x), \end{aligned}$$

where B is given as before by,

$$B = \frac{\beta p_{\ell h} z_h}{1 - \beta(1 - p_{\ell h})}.$$

and $\tilde{A}$ is given by,

$$\tilde{A} = \frac{\beta \ln(\delta)}{1 - \beta(1 - p_{\ell h})}[p_{\ell h} z_h + (1 - p_{\ell h}) z_\ell].$$

Observe that the coefficient B is the same but A is distinct from $\tilde{A}$.

Let $\tilde{x}$ be such that,

$$\ln(\tilde{x}) = \frac{\tilde{A}}{B - z_\ell}.$$

Assume that the rule is such that for $x \leq \tilde{x}$, it is always optimal to eat the cake. This requires that,

$$v(x, z_\ell) = z_\ell \ln(x).$$

On the other hand, if $x \geq \tilde{x}$ we will require that it is optimal not to eat the cake.

1. For all $x \leq \tilde{x}$: $z_\ell \ln(x) \geq \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h}) z_\ell \ln(\delta x)]$. This is equivalent to the condition that for all $x \leq \tilde{x}$: $z_\ell \ln(x) \geq \tilde{A} + B \ln(x)$. In order for this to be valid, we need that $B > z_\ell$.

2. For all $x \geq \tilde{x}$: $z_\ell \ln(x) \leq \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h}) v(\delta x)]$. This is easy, as $x \geq \tilde{x}$ we have that $z_\ell \ln(x) \leq \tilde{A} + B \ln(x)$ so,

$$\begin{aligned} z_\ell \ln(x) &\leq \tilde{A} + B \ln(x), \\ \leftrightarrow z_\ell \ln(x) &\leq \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h}) z_\ell \ln(\delta x))], \\ \rightarrow z_\ell \ln(x) &\leq \beta[p_{\ell h} z_h \ln(\delta x) + (1 - p_{\ell h}) v(\delta x, z_\ell))]. \end{aligned}$$

The last line follows from the fact that $v(\delta x) \geq z_\ell \ln(\delta x)$.

Lemma 15. *If $B > z_\ell$ then there is a unique point $\tilde{x}$ where*

$$\ln(\tilde{x}) = \frac{\tilde{A}}{B - z_\ell},$$

When $z = z_\ell$ then it is optimal to wait when $x \geq \tilde{x}$ and optimal to eat the cake when $x \leq \tilde{x}$.

Optimal stopping problems

Optimal stopping problems are a special class of problems in where the discrete choice is a single decision to put an end to an ongoing problem.[57]

As a first example, let p_t be the price of a given stock in period t. Assume that $p_{t+1} = p_t + z_{t+1}$ where z_t is an independently and identically distributed random variable with distribution F and mean zero. Suppose you have an option to buy the stock at a price c. Your problem then is,

$$v(p) = \max\{p - c, \beta \int v(p+z)F(dz)\}.$$

The first option to buy the stock at price c gives a benefit of $p - c$. If the stock is not bought, then the expected value of the next period is given by $\beta \int v(p+z)F(dz)$.

[57] For example, a student has to decide when to give up trying to solve a homework problem. A firms decides when to leave an industry, a firm decides when to stop working on the development of a new product or an unemployed worker has to decide when to accept a job from a sequence of offers.

Lemma 16. *$v(p) - p$ is non-increasing in p, $v(p)$ is increasing in p.*

Proof. Consider the contraction mapping.

$$(Tv)(p) = \max\{p - c, \beta \int v(p+z)F(dz)\}.$$

Let $v(p) - p$ be non-increasing in p. It suffices to show that $(Tv)(p) - p$ is non-increasing in p. We have that,

$$(Tv)(p) - p = \max\{-c, \beta \int v(p+z)F(dz) - p - \int zF(dz)\}.$$

The last term can be subtracted as $\int zF(dz) = 0$. Then,

$$(Tv)(p) - p = \max\{-c, \beta \int [v(p+z) - (p+z)]F(dz)\}.$$

Now $v(p+z) - (p+z)$ is decreasing in p by assumption. So the right hand side is non-increasing in p.

For the second, if $v(p)$ is increasing in p, then

$$(Tv)(p) = \max\{p - c, \beta \int v(p+z)F(dz)\},$$

which is increasing in p as the right hand is also. □

It is optimal to exercise the option today if,

$$p - c \geq \beta \int v(p+z)F(dz).$$

Lemma 17. *There is a p^* so it is optimal to buy the stock if $p \geq p^*$ and it is optimal to wait if $p \leq p^*$.*

Proof. Let p^* be the smallest value of p that satisfies the equation $v(p^*) - p^* = -c$. The left hand side is decreasing in p so there is such a value. Let $p < p^*$, then

$$\begin{aligned} v(p) - p &> -c, \\ \leftrightarrow v(p) &\geq p - c. \end{aligned}$$

This means that $v(p) = \beta \int v(p+z)F(dz)$ which means that it is optimal to wait. Now, if $p \geq p^*$ then $v(p) - p \leq -c$ so $v(p) \leq p - c$, which gives $v(p) = p - c$ so it is optimal to buy the stock. $\square$

When $p = p^*$ payoffs from buying and not buying are equal and,

$$p^* = c + \beta \int v(p+z)F(dz),$$

So the cutoff price is composed of two components, the cost of exercising the option and the option value of waiting.

Consider an agent that visits stores at a rate of one per period. Then given that the price quoted in the current period is p, the individual can choose to stop now and purchase the good or go to the next store. If he stops, he gets $u - p$ where u is the value of the good bought. If he continuous, he enters the next period as an active searcher. The Bellman equation is,

$$v(p) = \max\{u - p; -\beta c + \beta \int_0^\infty v(p')F(dp')\}.$$

Observe that the second term $-\beta c + \beta \int_0^\infty v(p')F(dp') = \bar{v}$ is independent of the current price p as we assumed that prices are i.i.d. drawn. The first term is declining in p so there is a unique value p^* where $u - p^* = -\beta c + \beta \int_0^\infty v(p')F(dp')$. From this, it follows that $\bar{v} = u - p^*$.

Any price greater than p^* induces further search while any value below p^* let's the agent buy the good. We have that,

$$\begin{aligned} u - p^* &= -\beta c + \beta \int_0^{p^*} v(p')F(dp') + \beta \int_{p^*}^\infty v(p')F(dp'), \\ &= -\beta c + \beta \int_0^{p^*} (u - p')F(dp') + \beta \int_{p^*}^\infty (u - p^*)F(dp'), \\ &= -\beta c + \beta(u - p^*) + \beta \int_0^{p^*} (p' - p^*)F(dp'). \end{aligned}$$

So,

$$p^* = u - \frac{\beta}{1-\beta}\left[-c + \int_0^{p^*} (p' - p^*)F(dp')\right]$$

This is the fundamental reservation price equation of the problem. The first term gives the immediate benefit of purchasing. The second term gives the option value of waiting.