MATH-S400

Assignment 4: Fixed points

1. Let $a, b \in \mathbb{R}$ with $a < b$ and consider a C^1 function $f : [a, b] \to [a, b]$ (C^1 means continuously differentiable). Show that if for some $x \in [a, b]$ $f'(x) > 1$, then f is not a contraction mapping on $[a, b]$ (here $f'(x)$ is the derivative of f at x). (hint: remember that $f'(x)$ is the limit of $\frac{f(x+\varepsilon_t)-f(x)}{\varepsilon_t}$ for a sequence $\varepsilon_t \to 0$)

2. (change to Markov transition matrix, columns sum to one) For any $n \in \mathbb{N}$ a $n \times n$ matrix A is called stochastic if for all $i, j \leq n$, $a_{i,j} \geq 0$ and $\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1$ (here $a_{i,j}$ is the entry of A in row i and column j). An example of a stochastic matrix is

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

   Prove that for any stochastic matrix A, there is an $\mathbf{x} \in \mathbb{R}^n_+$ such that $A\mathbf{x} = \mathbf{x}$ and $\sum_{j=1}^{n} (\mathbf{x})_j = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. On what domain is f a contraction mapping?

4. Consider a big house filled with similar shaped square rooms. There is one door in the building that leads to the outside. Additionally, every room in the house has either one or two doors (see figure below for an illustration). Prove that the house has an odd number of rooms with one door.