MATH-S400

Assignment 2

1. Consider the following theorem

   > If a subsequence of a Cauchy sequence in $\mathbb{R}^k$ converges to a vector $\mathbf{x} \in \mathbb{R}^k$, then the Cauchy sequence also converges to the same vector $\mathbf{x}$.

   Let, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^k$. Let,

   $$p \equiv (\mathbf{x}_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence.},$$
   $$q \equiv \exists \text{ subsequence } (\mathbf{x}_{\varphi(n)})_{n\in\mathbb{N}} : \mathbf{x}_{\varphi(n)} \stackrel{n}{\to} \mathbf{x},$$
   $$r \equiv \mathbf{x}_n \stackrel{n}{\to} \mathbf{x}.$$

   - Formulate the theorem in a logical formula using p, q and r.
   - Try to prove the theorem using a direct proof. Use the definitions of a Cauchy sequences, the convergence of a sequences, etc. to clearly state what you know and what you want to prove.
   - Try to prove the theorem using a proof by contrapositive. Again, clearly state what you know and what you want to prove.

2. Let $a_1, \dots, a_n, \dots$ be a sequence of numbers in $\mathbb{R}$ and define,

   $$x_n = \sum_{i=1}^{n} a_i,$$

   and

   $$y_n = \sum_{i=1}^{n} |a_i|.$$

   Show that if $(y_n)_{n\in\mathbb{N}}$ converges (i.e. is Cauchy), then $(x_n)_{n\in\mathbb{N}}$ also converges.

3. Show that if $0 \leq a_n \leq b_n$ and $x_n = \sum_{i=1}^{n} b_i$ converges, then $y_n = \sum_{i=1}^{n} a_n$ also converges. Use this result and the known non-convergence of the harmonic series $\sum_{i=1}^{n} 1/n$ to prove that $\sum_{i=1}^{n} 1/(\sqrt{n})$ also does not converge.

4. Demonstrate the reverse of the Bolzano-Weierstrass theorem:

   > If all sequences $(\mathbf{x})_{t\in\mathbb{N}}$ in a set $S \subseteq \mathbb{R}^n_+$ have a convergent subsequence with a limit in S, then S is compact (i.e. closed and bounded).

   - Assign formulas, $p, q, r, \ldots$, to different parts of this theorem and put the theorem as a logical statement (similar to exercise 1 above).

     Think about what type of proof you would use to prove the statement (direct proof, contrapositive, contradiction, induction, ...?) and prove the theorem.