Principles in Economics and Mathematics: the mathematical part

Bram De Rock

Practicalities about me

- Bram De Rock
- Office: R.42.6.218
- E-mail: bderock@ulb.ac.be
- Phone: 02 650 4214

- Mathematician who is doing research in Economics
- Homepage: http://www.revealedpreferences.org/bram.php

Practicalities about the course

- 12 hours on the mathematical part
- Micael Castanheira: 12 hours on the economics part
- Slides are available at MySBS and on http://mathecosolvay.com/spma/
- Schedule
  - Tuesday 17/9 and 24/9, 18.00-21.00, R42.2.107
  - Wednesday 18/9, 18.00-21.00, R42.2.103
  - Thursday 26/9, 18.00-21.00, R42.2.107
- Course evaluation
  - Written exam in the beginning of November to verify if you can apply the concepts discussed in class
  - Compulsory for students in *Financial Markets*
  - On a voluntary basis for students in *Quantitative Finance*

Course objectives and content

- Refresh some useful concepts needed in your other coursework
  - No thorough or coherent study
  - Interested student: see references for relevant material
- Content:
  1. Calculus (derivatives, optimization, concavity)
  2. Linear algebra (solving system of linear equations, matrices, linear (in)dependence)
  3. Fundamentals on probability (probability and cumulative distributions, expectations of a random variable, correlation)

References

- Chiang, A.C. and K. Wainwright, "*Fundamental Methods of Mathematical Economics*", Economic series, McGraw-Hill.
- Green, W.H., "*Econometric Analysis, Seventh Edition*", Pearson Education limited.
- Luderer, B., V. Nollau and K. Vetters, "*Mathematical Formulas for Economists*", Springer, New York. ULB-link
- Simon, C.P. and L. Blume "*Mathematics for Economists*", Norton & Company, New York.
- Sydsaeter, K., A. Strom and P. Berck, "*Economists' Mathematical Manual*", Springer, New York. ULB-link

Outline

Role of functions

- Calculus = “the study of functions”
- Functions allow to exploit mathematical tools in Economics
- E.g. make consumption decisions
  - max $U(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 = Y$
  - Characterization: $x_1 = f(p_1, p_1, Y)$
  - Econometrics: estimate f
  - Allows to model/predict consumption behavior
- Warning about identification
  - Causality: what is driving what?
  - Functional structure: what is driving the result?
  - Does the model allow to identify

Derivatives

- Marginal changes are important in Economics
  - The impact of a infinitesimally small change of one of the variables
  - Comparative statistics: what is the impact of a price change?
  - Optimization: what is the optimal consumption bundle?
- Marginal changes are mostly studied by taking derivatives
- Characterizing the impact depends on the function
  - $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^k : (x_1, \ldots, x_n) \mapsto (y_1, \ldots y_k) = f(x_1, \ldots, x_n)$
  - We will always take $k = 1$
  - First look at $n = 1$ and then generalize
  - Note: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Outline

Functions of one variable: $f : D \subseteq \mathbb{R} \to \mathbb{R}$

$$\frac{df}{dx} = f' = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Limit of quotient of differences
- If it exists, then it is called the derivative
- f' is again a function
- E.g. $f(x) = 3x^2 - 4$
- E.g. discontinuous functions, border of domain, $f(x) = |x|$

Some important derivatives and rules

Let us abstract from specifying the domain D and assume that $c, n \in \mathbb{R}_0$

- If $f(x) = c$, then $f'(x) = 0$
- If $f(x) = cx^n$, then $f'(x) = ncx^{n-1}$
- If $f(x) = ce^x$, then $f'(x) = ce^x$
- If $f(x) = c\ln(x)$, then $f(x) = c\frac{1}{x}$
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \neq f'(x)g'(x)$
- $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Application: the link with marginal changes

- By definition it is the limit of changes
- Slope of the tangent line
  - Increasing or decreasing function (and thus impact)
  - Does the inverse function exist?
- First order approximation in some point c
  - Based on expression for the tangent line in c
  - $f(c + \Delta x) \approx f(c) + f'(c)(\Delta x)$
  - More general approximation: Taylor expansion

Application: elasticities

- The elasticity of f in x: $\frac{f'(x)x}{f(x)}$
- The limit of the quotient of changes in terms of percentage
  - Percentage change of the function: $\frac{f(x+\Delta x)-f(x)}{f(x)}$
  - Percentage change of the variable: $\frac{\Delta x}{x}$
  - Quotient: $\frac{f(x+\Delta x)-f(x)}{\Delta x} \frac{x}{f(x)}$
- Is a unit independent informative number
  - E.g. the (price) elasticity of demand

Application: comparative statics for a simple market model

- Demand: $Q = 10 - 4P$
- Supply: $Q = 2 + \alpha P$
- $P^* = \frac{8}{4+\alpha}$ and $Q^* = \frac{8+10\alpha}{4+\alpha}$
- $\frac{dP^*}{d\alpha} = \frac{-8}{(4+\alpha)^2}$ and $\frac{dQ^*}{d\alpha} = \frac{32}{(4+\alpha)^2}$
- The elasticity of demand is $\frac{-4P}{10-4P}$

Some exercises

- Compute the derivative of the following functions (defined on $\mathbb{R}^+$)
  - $f(x) = 17x^2 + 5x + 7$
  - $f(x) = -\sqrt{x} + 3$
  - $f(x) = \frac{1}{x^2}$
  - $f(x) = 17x^2 e^x$
  - $f(x) = \frac{xln(x)}{x^2-4}$
- Let $f(x) : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 + 5x$.
  - Determine on which region f is increasing
  - Is f invertible?
  - Approximate f in 1 and derive an expression for the approximation error
  - Compute the elasticity in 3 and 5

The chain rule

- Often we have to combine functions
  - If $z = f(y)$ and $y = g(x)$, then $z = h(x) = f(g(x))$
- We have to be careful with the derivative
- A small change in x causes a chain reaction
  - It changes y and this in turn changes z
- That is why $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x)$
  - Can easily be generalized to compositions of more than two functions
  - $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{du}\cdots\frac{dv}{dx}$
- E.g. if $h(x) = e^{x^2}$, then $h'(x) = e^{x^2}2x$

Higher order derivatives

- The derivative is again a function of which we can take derivatives
- Higher order derivatives describe the changes of the changes
- Notation
  - $f''(x)$ or more generally $f^{(n)}(x)$
  - $\frac{d}{dx}(\frac{df}{dx})$ or more generally $\frac{d^n}{dx^n}f(x)$
- E.g. if $f(x) = 5x^3 + 2x$, then $f'''(x) = f^{(3)}(x) = 30$

Application: concave and convex functions

$$f : D \subset \mathbb{R}^n \to \mathbb{R}$$

- f is concave
  - $\forall x, y \in D, \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$
  - If $n = 1, \forall x \in D : f''(x) \leq 0$
- f is convex
  - $\forall x, y \in D, \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
  - If $n = 1, \forall x \in D : f''(x) \geq 0$

Application: concave and convex functions

- Very popular and convenient assumptions in Economics
  - E.g. optimization
- Sometimes intuitive interpretation
  - E.g. risk-neutral, -loving, -averse
- Don't be confused with a convex set
  - S is a set $\Leftrightarrow \forall x, y \in S, \forall \lambda \in [0,1] : \lambda x + (1-\lambda)y \in S$

Some exercises

- Compute the first and second order derivative of the following functions (defined on $\mathbb{R}^+$)
  - $f(x) = -\pi$
  - $f(x) = -\sqrt{5x} + 3$
  - $f(x) = e^{-3x}$
  - $f(x) = \ln(5x)$
  - $f(x) = x^3 - 6x^2 + 17$
- Determine which of these functions are concave or convex

Outline

Functions of more than one variable: $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

- Same applications in mind but now several variables
  - E.g. what is the marginal impact of changing x_1, while controlling for other variables?
- Look at the partial impact: partial derivatives
  - $\frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) = f_{x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i}$
  - Same interpretation as before, but now fixing remaining variables
- E.g. $f(x_1, x_2, x_3) = 2x_1^2 x_2 - 5x_3$
  - $\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = 4x_1 x_2$
  - $\frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = 2x_1^2$
  - $\frac{\partial}{\partial x_3} f(x_1, x_2, x_3) = -5$

Partial derivative

- Geometric interpretation: slope of tangent line in the x_i direction
- Same rules hold
- Higher order derivatives
  - $\frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$
  - $\frac{\partial^2}{\partial x_i x_j} f(x_1, \ldots, x_n) = \frac{\partial^2}{\partial x_j x_i} f(x_1, \ldots, x_n)$
  - E.g. $\frac{\partial^2}{\partial x_1^2} f(x_1, x_2, x_3) = 4x_2$ and $\frac{\partial^2}{\partial x_1 x_3} f(x_1, x_2, x_3) = 0$

Some remarks

- Gradient: $\nabla f(x_1, \ldots, x_n) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$
- Chain rule: special case
  - $x_1 = g_1(t), \ldots, x_n = g_n(t)$ and $f(x_1, \ldots, x_n)$
  - $h(t) = f(x_1, \ldots, x_n) = f(g_1(t), \ldots, g_n(t))$
  - $\frac{dh(t)}{dt} = h'(t) = \frac{\partial f(x_1,\ldots,x_n)}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f(x_1,\ldots,x_n)}{\partial x_n} \frac{dx_n}{dt}$
  - E.g. $f(x_1, x_2) = x_1 x_2$, $g_1(t) = e^t$ and $g_2(t) = t^2$
  - $h'(t) = e^t t^2 + e^t 2t$

Some remarks

- Slope of indifference curve of $f(x_1, x_2)$
  - Indifference curve: all (x_1, x_2) for which $f(x_1, x_2) = C$ (with C some give number)
  - Implicit function theorem: $f(x_1, g(x_1)) = C$
  - $\frac{\partial}{\partial x_1} f(x_1, x_2) + \frac{\partial}{\partial x_2} f(x_1, x_2) \frac{dg}{dx_1} = 0$
  - Slope $= -\frac{\frac{\partial}{\partial x_1} f(x_1, x_2)}{\frac{\partial}{\partial x_2} f(x_1, x_2)}$

Some exercises

- Compute the gradient and all second order partial derivatives for the following functions (defined on $\mathbb{R}^+$)
  - $f(x_1, x_2) = x_1^2 - 2x_1x_2 + 3x_2^2$
  - $f(x_1, x_2) = \ln(x_1x_2)$
  - $f(x_1, x_2, x_3) = e^{x_1+2x_2} - 3x_1x_3$
- Compute the marginal rate of substitution for the utility function $U(x_1, x_2) = x_1^{\alpha}x_2^{\beta}$

Outline

1. Introduction
2. Calculus
   - Motivation
   - Functions of one variable
   - Functions of more than one variable
   - Optimization
3. Linear algebra
4. Fundamentals of probability theory

Optimization: important use of derivatives

- Many models in economics entail optimizing behavior
    - Maximize/Minimize objective subject to constraints
- Characterize the points that solve these models
- Note on Mathematics vs Economics
    - Profit = Revenue - Cost
    - Marginal revenue = marginal cost
    - Marginal profit = zero

Optimization: formal problem

$$\max / \min f(x_1, \ldots, x_n)$$

$$s.t.$$

$$g_1(x_1, \ldots, x_n) = c_1$$

$$\ldots$$

$$g_m(x_1, \ldots, x_n) = c_m$$

$$x_1, \ \ldots x_n \geq 0$$

- Inequality constraints are also possible
- Kuhn-Tucker conditions

Necessity and sufficiency

- Necessary conditions based on first order derivatives
  - Local candidate for an optimum
- Sufficient conditions based on second order derivatives
- Necessary condition is sufficient if
  - The constraints are convex functions
  - E.g. no constraints, linear constraints, . . .
  - The objective function is concave: global maximum is obtained
  - The objective function is convex: global minimum is obtained
  - Often the “real” motivation in Economics

Necessary conditions

1. Free optimization
   - No constraints
   - $f'(x^*) = 0$ if $n = 1$
   - $\frac{\partial}{\partial x_i} f(x_1^*, \ldots, x_n^*) = 0$ for $i = 1, \ldots, n$
   - Intuitive given our geometric interpretation

Necessary conditions

2. Optimization with positivity constraints
   - No g_i constraints
   - On the boundary extra optima are possible
   - Often ignored: interior solutions
   - $x_i^* \geq 0$ for $i = 1, \ldots, n$
   - $x_i^* \frac{\partial}{\partial x_i} f(x_1^*, \ldots, x_n^*) = 0$ for $i = 1, \ldots, n$
   - $\frac{\partial}{\partial x_i} f(x_1^*, \ldots, x_n^*) \leq 0$ for all $i = 1, \ldots, n$ simultaneously OR
     $\frac{\partial}{\partial x_i} f(x_1^*, \ldots, x_n^*) \geq 0$ for all $i = 1, \ldots, n$ simultaneously

Necessary conditions

3. Constrained optimization without positivity constraints
   - Define Lagrangian: $L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) - \lambda_1(g_1(x_1, \ldots, x_n)) - \cdots - \lambda_m(g_m(x_1, \ldots, x_n))$
   - $\frac{\partial}{\partial x_i} L(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*) = 0$ for all $i = 1, \ldots, n$
   - $\frac{\partial}{\partial \lambda_j} L(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*) = 0$ for all $j = 1, \ldots, m$
   - Alternatively: $\nabla f(x_1^*, \ldots, x_n^*) = \lambda_1^* \nabla g_1(x_1^*, \ldots, x_n^*) + \cdots + \lambda_m^* \nabla g_m(x_1^*, \ldots, x_n^*)$
   - Some intuition: geometric interpretation
   - Lagrange multiplier = shadow price

Application: utility maximization

$$\max U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \;\; s.t. \;\; p_1 x_1 + p_2 x_2 = Y$$

- $L(x_1, x_2, \lambda_1) = x_1^{\alpha} x_2^{1-\alpha} - \lambda_1 (p_1 x_1 + p_2 x_2 - Y)$
- $\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha - 1} x_2^{1-\alpha} - \lambda_1 p_1 = 0$
- $\frac{\partial L}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda_1 p_2 = 0$
- $\frac{\partial L}{\partial \lambda_1} = p_1 x_1 + p_2 x_2 - Y = 0$
- $x_1^* = \frac{\alpha Y}{p_1}, x_2^* = \frac{(1-\alpha) Y}{p_2}$ and $\lambda_1^* = (\frac{\alpha}{p_1})^{\alpha} (\frac{(1-\alpha)}{p_2})^{(1-\alpha)}$

Some exercises

- Find the optima for the following problems
  - $\max / \min x^3 - 12x^2 + 36x + 8$
  - $\max / \min x_1^3 - x_2^3 + 9x_1x_2$
  - $\min 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 - 15x_1$
  - $\max x_1x_2$ s.t. $x_1 + 4x_2 = 16$
  - $\max yz + xz$ s.t. $y^2 + z^2 = 1$ and $xz = 3$
- Add positivity constraints to the above unconstrained problems and do the same

Outline

Motivation

- Matrices allow to formalize notation
- Useful in solving system of linear equations
- Useful in deriving estimators in econometrics
- Allows us to make the link with vector spaces

Outline

Matrices

$$A = (a_{ij})_{i=1,\ldots,n; j=1,\ldots,m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

- $a_{ij} \in \mathbb{R}$ and $A \in \mathbb{R}^{n \times m}$
  - n rows and m columns
- Square matrix if $n = m$
- Notable square matrices
  - Symmetric matrix: $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$
  - Diagonal matrix: $a_{ij} = 0$ for all $i, j = 1, \ldots, n$ and $i \neq j$
  - Triangular matrix: only non-zero elements above (or below) the diagonal

Matrix manipulations

Let $A, B \in \mathbb{R}^{n \times m}$ and $k \in \mathbb{R}$

- Equality: $A = B \Leftrightarrow a_{ij} = b_{ij}$ for all $i, j = 1, \ldots, n$
- Scalar multiplication: $kA = (ka_{ij})_{i=1,\ldots,n; j=1,\ldots,m}$
- Addition: $A \pm B = (a_{ij} \pm b_{ij})_{i=1,\ldots,n; j=1,\ldots,m}$
  - Dimensions must be equal
- Transposition: $A' = A^t = (a_{ji})_{j=1,\ldots,m; i=1,\ldots,n}$
  - $A \in \mathbb{R}^{n \times m}$ and $A^t \in \mathbb{R}^{m \times n}$
  - $(A \pm B)^t = A^t \pm B^t$
  - $(kA)^t = kA^t$
  - $(A^t)^t = A$

Matrix multiplication

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$

- $AB = (\sum_{h=1}^{m} a_{ih} b_{hj})_{i=1,\ldots,n;j=1,\ldots,k}$
- Multiply the row vector of A with the column vector of B
  - Aside: scalar/inner product and norm of vectors
  - Orthogonal vectors
- Number of columns of A must be equal to number of rows of B
- $AB \neq BA$, even if both are square matrices
- $(AB)^t = B^t A^t$

Example

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

- $A^t = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B^t \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$
- $AB = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$
- $AC = \begin{pmatrix} 5 & 6 & 7 \\ 4 & 4 & 4 \end{pmatrix}$
- $(AB)^t = B^t A^t = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Exercises

- Let $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 4 & 0 & -3 & 1 \end{pmatrix}$
  - Compute $-3C$, $A + B$, $A - D$ and D^t
  - Compute AB, BA, AC, CA, AD and DA
- Let A be a symmetric matrix, show then that $A^t = A$
- A square matrix A is called idempotent if $A^2 = A$
  - Verify which of the above matrices are idempotent
  - Find the value of α that makes the following matrix idempotent: $\begin{pmatrix} -1 & 2 \\ \alpha & 2 \end{pmatrix}$

Two numbers associated to square matrices: trace

Let $A, B, C \in \mathbb{R}^{n\times n}$

- Trace(A) $= tr(A) = \sum_{i=1}^{n} a_{ii}$
- Used in econometrics
- Properties
  - $tr(A^t) = tr(A)$
  - $tr(A + B) = tr(A) + tr(B)$
  - $tr(cA) = ctr(A)$ for any $c \in \mathbb{R}$
  - $tr(AB) = tr(BA)$
  - $tr(ABC) = tr(BCA) = tr(CAB) \neq tr(ACB)(= tr(BAC) = tr(CBA))$

Two numbers associated to square matrices: trace

Example

- Let $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$
- Then $AB = \begin{pmatrix} 21 & 0 \\ -7 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 2 \\ 10 & 20 \end{pmatrix}$ and $A + B = \begin{pmatrix} 4 & 8 \\ 2 & -3 \end{pmatrix}$
- $tr(A) = 1$, $tr(B) = 0$ and $tr(A + B) = 1$
- $tr(AB) = 21 = tr(BA)$

Two numbers associated to square matrices: determinant

Let $A \in \mathbb{R}^{n \times n}$

- If $n = 1$, then $\det(A) = a_{11}$
- If $n = 2$, then
  $\det(A) = a_{11}a_{22} - a_{12}a_{21} = a_{11}\det(a_{22}) - a_{12}\det(a_{21})$
- If $n = 3$, then $\det(A) = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} -$
  $a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
- Can be generalized to any n
- Works with columns too

Two numbers associated to square matrices: determinant

Let $A, B \in \mathbb{R}^{n \times n}$

- $\det(A^t) = \det(A)$
- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(cA) = c\det(A)$ for any $c \in \mathbb{R}$
- $\det(AB) = \det(BA)$
- A is non-singular (or regular) if A^{-1} exists
    - I.e. $AA^{-1} = A^{-1}A = I_n$
    - I_n is a diagonal matrix with 1 on the diagonal
    - Does not always exist
    - $\det(A) \neq 0$

Two numbers associated to square matrices: determinant

Example

- Let $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$
- Then $AB = \begin{pmatrix} 21 & 0 \\ -7 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 2 \\ 10 & 20 \end{pmatrix}$ and $A + B = \begin{pmatrix} 4 & 8 \\ 2 & -3 \end{pmatrix}$
- $\det(A) = 0$, $\det(B) = -7$ and $\det(A + B) = -28$
- $\det(AB) = 0 = \det(BA)$

Exercises

- Let $A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
  - Compute $tr(A)$ and $\det(-2A)$
- Show that for any triangular matrix A, we have that $\det(A)$ is equal to the product of the elements on the diagonal
- Let $A, B \in \mathbb{R}^{n \times n}$ and assume that B is non-singular
  - Show that $tr(B^{-1}AB) = tr(A)$
  - Show that $tr(B(B^t B)^{-1} B^t) = n$
- Let $A, B \in \mathbb{R}^{n \times n}$ be two non-singular matrices
  - Show that AB is then also invertible
  - Give an expression for $(AB)^{-1}$

Outline

A notion of vector spaces

- A set of vectors V is a vector space if
  - Addition of vectors is well-defined
  - $\forall a, b \in V : a + b \in V$
  - Scalar multiplication is well-defined
  - $\forall k \in \mathbb{R}, \forall a \in V : ka \in V$
- We can take linear combinations
  - $\forall k_1, k_2 \in \mathbb{R}, \forall a, b \in V : k_1 a + k_2 b \in V$
- E.g. $\mathbb{R}^2$ or more generally $\mathbb{R}^n$
- Counterexample $\mathbb{R}^2_+$

Linear (in)dependence

Let V be a vector space

- A set of vectors $v_1, \ldots, v_n \in V$ is *linear dependent* if one of the vectors can be written as a linear combination of the others
  - $\exists k_1, \ldots k_{n-1} \in \mathbb{R} : v_n = k_1 v_1 + \cdots + k_{n-1} v_{n-1}$
- A set of vectors are *linear independent* if they are not linear dependent
  - $\forall k_1, \ldots k_n \in \mathbb{R} : k_1 v_1 + \cdots + k_n v_n = 0 \Rightarrow k_1 = \cdots = k_n = 0$
- In a vector space of dimension n, the number of linear independent vectors cannot be higher than n

Linear (in)dependence

Example

- $\mathbb{R}^2$ is a vector space of dimension 2
- $v_1 = (1,0)$, $v_2 = (1,2)$, $v_3 = (-1,4)$ and $v_4 = (2,4)$
- $v_3 = -3v_1 + 2v_2$, so v_1, v_2, v_3 are linear dependent
- $v_4 = 2v_2$, so v_2, v_4 are linear dependent
- v_1, v_2 are linear independent
- v_3 is linear independent

Link with matrices: rank

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$

- The row rank of A is the maximal number of linear independent rows of A
- The column rank of A is the maximal number of linear independent columns of A
- Rank of A = column rank of A = row rank of A
- Properties
  - $rank(A) \leq \min(n, m)$
  - $rank(AB) \leq \min(rank(A), rank(B))$
  - $rank(A) = rank(A^t A) = rank(AA^t)$
  - If $n = m$, then A has maximal rank if and only if $\det(A) \neq 0$

Link with matrices: rank

Example

- Let $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$
- Then $AB = \begin{pmatrix} 21 & 0 \\ -7 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 2 \\ 10 & 20 \end{pmatrix}$
- $rank(A) = 1$ and $rank(B) = 2$
- $rank(AB) = 1$

Exercises

- Let $A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
  - Show in two ways that A has maximal rank
- Let $A, B \in \mathbb{R}^{n \times m}$
  - Show that there need not be any relation between $rank(A + B)$, $rank(A)$ and $rank(B)$
- Show that if A is invertible, then it needs to have a maximal rank

Outline

System of linear equations

- Linear equations in the unknowns $x_1, \ldots, x_m$
  - Not $x_1 x_2$, x_m^2,...
- Constraints hold with equality
  - Not $2x_1 + 5x_2 \leq 3$
- E.g. $\begin{cases} 2x_1 + 3x_2 - x_3 = 5 \\ -x_1 + 4x_2 + x_3 = 0 \end{cases}$

Using matrix notation

- m unknowns: $x_1, \ldots, x_m$
- n linear constraints: $a_{i1}x_1 + \cdots + a_{im}x_m = b_i$ with $a_{i1}, \ldots a_{im}, b_i \in \mathbb{R}$ and $i = 1, \ldots, n$
- $Ax = b$
    - $A = (a_{ij})_{i=1,\ldots,n, j=1,\ldots,m}$
    - $x = (x_i)_{i=1,\ldots,n}$
    - $b = (b_i)_{i=1,\ldots,n}$
- Homogeneous if $b_i = 0$ for all $i = 1, \ldots, n$

Solving these system

- Solve this by logical reasoning
  - Eliminate or substitute variables
  - Can also be used for non-linear systems of equations
  - Can be cumbersome for larger systems
- Use matrix notation
  - Gaussian elimination of the augmented matrix $(A|b)$
  - Can be programmed
  - Only for systems of linear equations
  - Theoretical statements are possible

Example

$$\begin{cases} -x_1 + 4x_2 = 0 \\ 2x_1 + 3x_2 = 5 \end{cases}$$

- $x_1 = 4x_2 \Rightarrow 11x_2 = 5 \Rightarrow x_2 = \frac{5}{11}$ and $x_1 = \frac{20}{11}$
- $\begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$
- Take linear combinations of rows of the augmented matrix
  - Is the same as taking linear combinations of the equations
  - $\begin{pmatrix} -1 & 4 & |0 \\ 2 & 3 & |5 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 4 & | & 0 \\ 0 & 11 & | & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & | & \frac{-20}{11} \\ 0 & 11 & | & 5 \end{pmatrix}$

Theoretical results

Consider the linear system $Ax = b$ with $A \in \mathbb{R}^{n \times m}$

- This system has a solution if and only if $rank(A) = rank(A|b)$
  - $rank(A) \leq rank(A|b)$ by definition
  - Is the (column) vector b a linear combination of the column vectors of A?
  - If $rank(A) < rank(A|B)$, the answer is no
  - If $rank(A) = rank(A|B)$, the answer is yes
- The solution is unique if $rank(A) = rank(A|B) = m$
  - $n \geq m$
- There are ∞ many solutions if $rank(A) = rank(A|B) < m$
  - $n < m$ or too many constraints are 'redundant'

Exercises

- Solve the following systems of linear equations
  - $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 1 \end{cases}$ and $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 3x_1 + x_2 + x_3 = 0 \\ 4x_1 + 2x_3 = -1 \end{cases}$
- For which values of k does the following system of linear equations have a unique solution?
  - $\begin{cases} x_1 + x_2 = 1 \\ x_1 - kx_2 = 1 \end{cases}$
- Consider the linear system $Ax = b$ with $A \in \mathbb{R}^{n \times n}$
  - Show that this system has a unique solution if and only if A is invertible
  - Give a formula for this unique solution
- Show that homogeneous systems of linear equations always have a (possibly non-unique) solution

Outline

- To be continued